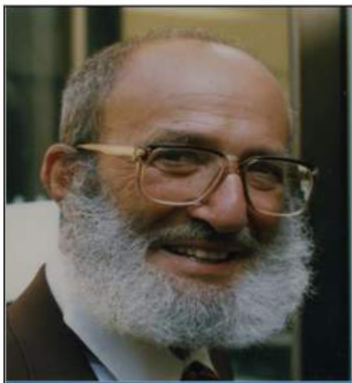

The only way to learn mathematics is to do mathematics.

— Paul Halmos —

THE LOGIC OF COMPOUND STATEMENS

I hear and I forget.
I see and I remember.
I do and I understand.
— Confucious —

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STATEMENTS

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A statement is a declarative sentence that is true or false but not both.

Example:

1) It will rain today.
2) Pune is capital city of Maharashtra.
3) 2 – 5 = 3.
4) $\cos^2\theta + \sin^2\theta = 1, for\ all\ real\ \theta.$
5) $x^2 - 3x - 18 = 0.$
6) Is office working today?
7) Don't shout.

COMPOUND STATEMENTS

Logical Connectives:

1) Negation
2) Conjunction
3) Disjunction

COMPOUND STATEMENTS

Negation:

Suppose p is any statement.

The negation of p, denoted by $\sim p$.

The sentence $\sim p$ is read "*not p*" or "*It is not the case that p*".

In some computer languages the symbol $\neg$ is used in place of $\sim$.

Example:

p: *It is sunny*.

$\sim p$: *It is not the case that it is sunny*. OR *It is not sunny*.

COMPOUND STATEMENTS

Conjunction:

Suppose p and q are any two statements.

The conjunction of p and q, denoted by $p \wedge q$, is the compound statement "p *and* q".

Example:

p: *Delhi is capital of India.*

q: *Moscow is capital of Russia.*

$p \wedge q$: *Delhi is capital of India and Moscow is capital of Russia.*

COMPOUND STATEMENTS

Disjunction:

Suppose p and q are any two statements.

The disjunction of p and q, denoted by $p \vee q$, is the compound statement "p *or* q".

Example:

p: *Delhi is capital of India.*

q: *Moscow is capital of Russia.*

$p \vee q$: *Delhi is capital of India or Moscow is capital of Russia.*

COMPOUND STATEMENTS

But and Neither-Nor:

p *but* q	means	p *and* q
neither p *nor* q	means	$\sim p$ *and* $\sim q$

Example:

Let p: *It is hot* and q: *It is sunny*.

It is not hot but it is sunny.

Symbolically: $\sim p \wedge q$

It is neither hot nor sunny.

Symbolically: $\sim p \wedge \sim q$

COMPOUND STATEMENTS

And, or and Inequalities:

If $x. a$ *and* b are particular real numbers, then

$x \leq a$	means	$x < a$ *or* $x = a$
$a \leq x \leq b$	means	$a \leq x$ *and* $x \leq b$

Example:

Let p, q and r symbolize "$0 < x$“, "$x < 3$“ and "$x = 3$“.

Then i) $x \leq 3$, symbolically: $q \vee r$

ii) $0 < x < 3$, symbolically: $p \wedge q$

iii) $0 < x \leq 3$, symbolically: $p \wedge (q \vee r)$

EXERCISES

Write the statements in 6–9 in symbolic form using the symbols $\sim$, $\vee$, and $\wedge$ and the indicated letters to represent component statements.

6. Let s = "stocks are increasing" and i = "interest rates are steady."
   a. Stocks are increasing but interest rates are steady.
   b. Neither are stocks increasing nor are interest rates steady.

7. Juan is a math major but not a computer science major. (m = "Juan is a math major," c = "Juan is a computer science major")

EXERCISES

8. Let h = "John is healthy," w = "John is wealthy," and s = "John is wise."
   a. John is healthy and wealthy but not wise.
   b. John is not wealthy but he is healthy and wise.
   c. John is neither healthy, wealthy, nor wise.
   d. John is neither wealthy nor wise, but he is healthy.
   e. John is wealthy, but he is not both healthy and wise.

EXERCISES

9. Either this polynomial has degree 2 or it has degree 3 but not both. ($n =$ "This polynomial has degree 2," $k =$ "This polynomial has degree 3")

EXERCISES

10. Let p be the statement "DATAENDFLAG is off," q the statement "ERROR equals 0," and r the statement "SUM is less than 1,000." Express the following sentences in symbolic notation.
    a. DATAENDFLAG is off, ERROR equals 0, and SUM is less than 1,000.
    b. DATAENDFLAG is off but ERROR is not equal to 0.
    c. DATAENDFLAG is off; however, ERROR is not 0 or SUM is greater than or equal to 1,000.
    d. DATAENDFLAG is on and ERROR equals 0 but SUM is greater than or equal to 1,000.
    e. Either DATAENDFLAG is on or it is the case that both ERROR equals 0 and SUM is less than 1,000.

TRUTH VALUES

If p is a statement and statement p is true then the statement p is said to have truth value "T",

and if statement p is false then the statement p is said to have truth value "F".

Truth Table: A table that shows the truth value of a compound statement for every truth value of its component statements.

TRUTH TABLE OF $\sim p$

$\sim p$ has opposite truth value from p.

i.e. if p is true then $\sim p$ is false; if p is false then $\sim p$ is true.

Truth Table for $\sim p$

p	$\sim p$
T	F
F	T

TRUTH TABLE OF $p \wedge q$

The compound statement $p \wedge q$ is true when, and only when, both p and q are true.

Truth Table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

TRUTH TABLE OF $p \vee q$

The compound statement $p \vee q$ is false when, and only when, both p and q are false.

Truth Table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

TRUTH TABLE FOR EXCLUSIVE OR

When *"or"* is used in its exclusive sense, the statement *"p or q"* means *"p or q but not both"*.

Symbolically: $(p \vee q) \wedge \sim(p \wedge q)$

It is denoted by $p \oplus q$ or $p\ XOR\ q$.

Truth Table for *Exclusive Or:* $(p \vee q) \wedge \sim(p \wedge q)$

p	q	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$(p \vee q) \wedge \sim(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

TRUTH TABLES

Construct truth table for the statement form $(p \wedge q) \vee \sim r$.

p	q	r	$p \wedge q$	$\sim r$	$(p \wedge q) \vee \sim r$
T	T	T	T	F	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	T
F	T	T	F	F	F
F	T	F	F	T	T
F	F	T	F	F	F
F	F	F	F	T	T

TRUTH TABLES

Construct truth table for the following statement forms:

i) $\sim p \wedge q$

ii) $\sim(p \wedge q) \vee (p \vee q)$

iii) $p \wedge (q \wedge r)$

iv) $p \wedge (\sim q \vee r)$

LOGICAL EQUIVALENCE

Two *statement forms* are called logically equivalent if, and only if, they have identical truth values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms P and Q is denoted by writing $P \equiv Q$.

LOGICAL EQUIVALENCE

Double Negation Property: $\sim(\sim p) \equiv p$

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

p and $\sim(\sim p)$ always have the same truth values, so they are logically equivalent

LOGICAL EQUIVALENCE

Showing Non-equivalence:

$$\sim(p \wedge q) \not\equiv \sim p \wedge \sim q$$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$		$\sim p \wedge \sim q$
T	T	F	F	T	F		F
T	F	F	T	F	T	$\neq$	F
F	T	T	F	F	T	$\neq$	F
F	F	T	T	F	T		T

$\sim(p \wedge q)$ and $\sim p \wedge \sim q$ have different truth values in rows 2 and 3, so they are not logically equivalent

LOGICAL EQUIVALENCE

Determine whether the statement forms given below are logically equivalent. In each case, construct a truth table and include a sentence justifying your answer.

16. $p \vee (p \wedge q)$ and p
17. $\sim(p \wedge q)$ and $\sim p \wedge \sim q$
18. $p \vee \mathbf{t}$ and $\mathbf{t}$
19. $p \wedge \mathbf{t}$ and p
20. $p \wedge \mathbf{c}$ and $p \vee \mathbf{c}$
21. $(p \wedge q) \wedge r$ and $p \wedge (q \wedge r)$
22. $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$
23. $(p \wedge q) \vee r$ and $p \wedge (q \vee r)$
24. $(p \vee q) \vee (p \wedge r)$ and $(p \vee q) \wedge r$

NEGATION OF *AND* and *OR*

De Morgan's Laws:

i) The negation of an *and* statement is logically equivalent to the *or* statement in which each component is negated.

Symbolically: $\sim(p \wedge q) \equiv \sim p \vee \sim q$

ii) The negation of an *or* statement is logically equivalent to the *and* statement in which each component is negated.

Symbolically: $\sim(p \vee q) \equiv \sim p \wedge \sim q$

NEGATION OF *AND* and *OR*

Applying De Morgan's Laws:

Write negations for each of the following statements:

a) John is 6 feet tall and he weighs at least 200 pounds.

Ans: John is not 6 feet tall or he weighs less than 200 pounds.

b) The bus was late or Tom's watch was slow.

Ans: The bus was not late and Tom's watch was not slow.

NEGATION OF *AND* and *OR*

Use De Morgan's laws to write negations for each of the following statements:

25. Hal is a math major and Hal's sister is a computer science major.
26. Sam is an orange belt and Kate is a red belt.
27. The connector is loose or the machine is unplugged.
28. The units digit of 4^{67} is 4 or it is 6.
29. This computer program has a logical error in the first ten lines or it is being run with an incomplete data set.
30. The dollar is at an all-time high and the stock market is at a record low.
31. The train is late or my watch is fast.

NEGATION OF *AND* and *OR*

Inequalities and De Morgan's Laws:

Use De Morgan's laws to write the negation of $-1 < x \le 4$.

Solution: The given statement is equivalent to

$$-1 < x \quad and \quad x \le 4.$$

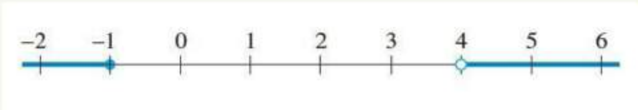
By De Morgan's laws, the negation is

$$-1 \nless x \quad or \quad x \nleq 4,$$

which is equivalent to

$$-1 \ge x \quad or \quad x > 4.$$

Pictorially,



NEGATION OF *AND* and *OR*

Assume x is a particular real number and use De Morgan's laws to write negations for the statements in 32–37.

32. $-2 < x < 7$

33. $-10 < x < 2$

34. $x < 2$ or $x > 5$

35. $x \le -1$ or $x > 1$

36. $1 > x \ge -3$

37. $0 > x \ge -7$

TAUTOLOGIES and CONTRADICTIONS

A tautology is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a tautology is a tautological statement.

A contradiction is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a contradiction is a contradictory statement.

TAUTOLOGIES and CONTRADICTIONS

Show that the statement form $p \vee \sim p$ is a tautology and that the statement form $p \wedge \sim p$ is a contradiction.

p	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F
F	T	T	F
		↑ all T's so $p \vee \sim p$ is a tautology	↑ all F's so $p \wedge \sim p$ is a contradiction

TAUTOLOGIES and CONTRADICTIONS

If t is a tautology and c is a contradiction, show that $p \wedge t \equiv p$ and $p \wedge c \equiv c$.

p	$\mathbf{t}$	$p \wedge \mathbf{t}$	p	$\mathbf{c}$	$p \wedge \mathbf{c}$
T	T	T	T	F	F
F	T	F	F	F	F

same truth values, so $p \wedge \mathbf{t} \equiv p$

same truth values, so $p \wedge \mathbf{c} \equiv \mathbf{c}$

TAUTOLOGIES and CONTRADICTIONS

Use truth tables to establish which of the statement forms in 40–43 are tautologies and which are contradictions.

40. $(p \wedge q) \vee (\sim p \vee (p \wedge \sim q))$

41. $(p \wedge \sim q) \wedge (\sim p \vee q)$

42. $((\sim p \wedge q) \wedge (q \wedge r)) \wedge \sim q$

43. $(\sim p \vee q) \vee (p \wedge \sim q)$

LOGICAL EQUIVALENCES

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1.	*Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2.	*Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3.	*Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4.	*Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5.	*Negation laws:*	$p \vee \sim p \equiv \mathbf{t}$	$p \wedge \sim p \equiv \mathbf{c}$
6.	*Double negative law:*	$\sim(\sim p) \equiv p$	
7.	*Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8.	*Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9.	*De Morgan's laws:*	$\sim(p \wedge q) \equiv \sim p \vee \sim q$	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
10.	*Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11.	*Negations of* $\mathbf{t}$ *and* $\mathbf{c}$*:*	$\sim \mathbf{t} \equiv \mathbf{c}$	$\sim \mathbf{c} \equiv \mathbf{t}$

LOGICAL EQUIVALENCES

Simplifying Statement Forms:

Verify the logical equivalence $\sim (\sim p \wedge q) \wedge (p \vee q) \equiv p$.

Solution:

$$
\begin{aligned}
\sim(\sim p \wedge q) \wedge (p \vee q) &\equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q) && \text{by De Morgan's laws} \\
&\equiv (p \vee \sim q) \wedge (p \vee q) && \text{by the double negative law} \\
&\equiv p \vee (\sim q \wedge q) && \text{by the distributive law} \\
&\equiv p \vee (q \wedge \sim q) && \text{by the commutative law for } \wedge \\
&\equiv p \vee \mathbf{c} && \text{by the negation law} \\
&\equiv p && \text{by the identity law.}
\end{aligned}
$$

LOGICAL EQUIVALENCES

In 48 and 49 below, a logical equivalence is derived from Theorem 2.1.1. Supply a reason for each step.

48. $(p \wedge \sim q) \vee (p \wedge q) \equiv p \wedge (\sim q \vee q)$ by (a) ____

$\equiv p \wedge (q \vee \sim q)$ by (b) ____

$\equiv p \wedge \mathbf{t}$ by (c) ____

$\equiv p$ by (d) ____

Therefore, $(p \wedge \sim q) \vee (p \wedge q) \equiv p$.

49. $(p \vee \sim q) \wedge (\sim p \vee \sim q)$

$\equiv (\sim q \vee p) \wedge (\sim q \vee \sim p)$ by (a) ____

$\equiv \sim q \vee (p \wedge \sim p)$ by (b) ____

$\equiv \sim q \vee \mathbf{c}$ by (c) ____

$\equiv \sim q$ by (d) ____

Therefore, $(p \vee \sim q) \wedge (\sim p \vee \sim q) \equiv \sim q$.

LOGICAL EQUIVALENCES

Use Theorem 2.1.1 to verify the logical equivalences in 50–54. Supply a reason for each step.

50. $(p \wedge \sim q) \vee p \equiv p$

51. $p \wedge (\sim q \vee p) \equiv p$

52. $\sim(p \vee \sim q) \vee (\sim p \wedge \sim q) \equiv \sim p$

53. $\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q) \equiv p$

54. $(p \wedge (\sim(\sim p \vee q))) \vee (p \wedge q) \equiv p$

CONDITIONAL STATEMENTS

If p and q are statement variables, the conditional of q by p is "If p then q" or "p implies q" and is denoted $p \to q$.

It is false when p is true and q is false; otherwise it is true. We call p the hypothesis (or antecedent) of the conditional and q the conclusion (or consequent).

Example:

If 4686 is divisible by 6, then 4686 is divisible by 3.

Hypothesis — Conclusion

Truth Table for $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

TRUTH TABLE

In expressions that include $\to$ as well as other logical operators such as $\wedge$, $\vee$ and $\sim$, the order of operations is that $\to$ is performed last. Thus, the order of operations is, $\sim$ is performed first, then $\wedge$ and $\vee$, and finally $\to$.

Example:

Construct a truth table for the statement form $p \vee \sim q \to \sim p$.

p	q	$\sim q$	$p \vee \sim q$	$\sim p$	$p \vee \sim q \to \sim p$
T	T	F	T	F	F
T	F	T	T	F	F
F	T	F	F	T	T
F	F	T	T	T	T

TRUTH TABLE

Construct truth tables for the statement forms in 5–11.

5. $\sim p \vee q \rightarrow \sim q$

6. $(p \vee q) \vee (\sim p \wedge q) \rightarrow q$

7. $p \wedge \sim q \rightarrow r$

8. $\sim p \vee q \rightarrow r$

9. $p \wedge \sim r \leftrightarrow q \vee r$

10. $(p \rightarrow r) \leftrightarrow (q \rightarrow r)$

11. $(p \rightarrow (q \rightarrow r)) \leftrightarrow ((p \wedge q) \rightarrow r)$

LOGICAL EQUIVALENCES

Division into Cases: $p \vee q \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$

p	q	r	$p \vee q$	$p \rightarrow r$	$q \rightarrow r$	$p \vee q \rightarrow r$	$(p \rightarrow r) \wedge (q \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

$p \vee q \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$ always have the same truth values, so they are logically equivalent ■

LOGICAL EQUIVALENCES

Representation of IF-THEN As OR:

$$p \longrightarrow q \equiv \sim p \vee q$$

Example:

Either you get to work on time or you are fired.

Let $\sim p$: You get to work on time, and q: You are fired.

Then p: You do not get to work on time.

The given statement in symbolic notation is, $\sim p \vee q$

And $\sim p \vee q \equiv p \longrightarrow q$

Then the equivalent if-then version, $p \longrightarrow q$, is

If you do not get to work on time, then you are fired.

LOGICAL EQUIVALENCES

The Negation of a Conditional Statement:

The negation of "if p then q" is logically equivalent to "p and not q."

Symbolically: $\sim(p \longrightarrow q) \equiv p \wedge \sim q$

Proof:

$\sim(p \longrightarrow q) \equiv \sim(\sim p \vee q)$ — *Since* $p \longrightarrow q \equiv \sim p \vee q$

$\equiv \sim(\sim p) \wedge \sim q$ — By De-Morgan's law

$\equiv p \wedge \sim q$ — By double negative law

13. Use truth tables to verify the following logical equivalences. Include a few words of explanation with your answers.

    a. $p \to q \equiv \sim p \vee q$ b. $\sim(p \to q) \equiv p \wedge \sim q$.

LOGICAL EQUIVALENCES

The Negation of a Conditional Statement:

Write negations for each of the following statements:

1) If my car is in the repair shop, then I cannot get to class.

Ans: My car is in the repair shop and I can get to class.

2) If Sara lives in Athens, then she lives in Greece.

Ans: Sara lives in Athens and she does not live in Greece.

CONTRAPOSITIVE

The contrapositive of a conditional statement of the form "If p then q" is "If $\sim q$ then $\sim p$".

Symbolically: The contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$.

Note: A conditional statement is logically equivalent to its contrapositive.

Write each of the following statements in its equivalent contrapositive form:

1) If Howard can swim across the lake, then Howard can swim to the island.

Ans: If Howard cannot swim to the island, then Howard cannot swim across the lake.

2) If today is Easter, then tomorrow is Monday.

Ans: If tomorrow is not Monday, then today is not Easter.

CONVERSE and INVERSE

Suppose a conditional statement of the form "If p then q" is given.

1) The converse is "If q then p."
2) The inverse is "If $\sim p$ then $\sim q$."

Symbolically:

The converse of $p \rightarrow q$ is $q \rightarrow p$,

and

the inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$.

Note:

1) A conditional statement and its converse are *not* logically equivalent.
2) A conditional statement and its inverse are *not* logically equivalent.
3) The converse and the inverse of a conditional statement are logically equivalent to each other.

CONVERSE and INVERSE

Write the converse and inverse of each of the following statements:

1) If Howard can swim across the lake, then Howard can swim to the island.

Ans: Converse: If Howard can swim to the island, then Howard can swim across the lake.

Inverse: If Howard cannot swim across the lake, then Howard cannot swim to the island.

2) If today is Easter, then tomorrow is Monday.

Ans: Converse: If tomorrow is Monday, then today is Easter.

Inverse: If today is not Easter, then tomorrow is not Monday.

EXERCISE

20. Write negations for each of the following statements. (Assume that all variables represent fixed quantities or entities, as appropriate.)

    a. If P is a square, then P is a rectangle.
    b. If today is New Year's Eve, then tomorrow is January.
    c. If the decimal expansion of r is terminating, then r is rational.
    d. If n is prime, then n is odd or n is 2.
    e. If x is nonnegative, then x is positive or x is 0.
    f. If Tom is Ann's father, then Jim is her uncle and Sue is her aunt.
    g. If n is divisible by 6, then n is divisible by 2 and n is divisible by 3.

EXERCISE

21. Suppose that p and q are statements so that $p \to q$ is false. Find the truth values of each of the following:

    a. $\sim p \to q$ b. $p \vee q$ c. $q \to p$

22. Write contrapositives for the statements of exercise 20.

23. Write the converse and inverse for each statement of exercise 20.

EXERCISE

Use truth tables to establish the truth of each statement in 24–27.

24. A conditional statement is not logically equivalent to its converse.

25. A conditional statement is not logically equivalent to its inverse.

26. A conditional statement and its contrapositive are logically equivalent to each other.

27. The converse and inverse of a conditional statement are logically equivalent to each other.

ONLY IF

If p and q are statements,

p only if q means "if not q then not p",

or, equivalently,

"*if* p then q".

Rewrite the following statement in if-then form in two ways, one of which is the contrapositive of the other.

John will break the world's record for the mile run only if he runs the mile in under four minutes.

Ans:

Version 1: If John does not run the mile in under four minutes, then he will not break the world's record.

Version 2: If John breaks the world's record, then he will have run the mile in under four minutes.

BICONDITIONAL

Given statement variables p and q, the biconditional of p and q is "p if, and only if, q" and is denoted $p \leftrightarrow q$.

It is true if both p and q have the same truth values and is false if p and q have opposite truth values.

The words *if and only if* are sometimes abbreviated iff.

Truth Table for $p \leftrightarrow q$

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

ORDER OF OPERATION

Order of Operations for Logical Operators

1. $\sim$ — Evaluate negations first.
2. $\wedge, \vee$ — Evaluate $\wedge$ and $\vee$ second. When both are present, parentheses may be needed.
3. $\rightarrow, \leftrightarrow$ — Evaluate $\rightarrow$ and $\leftrightarrow$ third. When both are present, parentheses may be needed.

BICONDITIONAL

Show that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$.

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

$p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$ always have the same truth values, so they are logically equivalent

BICONDITIONAL

Rewrite the following statement as a conjunction of two if-then statements:

This computer program is correct if, and only if, it produces correct answers for all possible sets of input data.

Ans: If this program is correct, then it produces the correct answers for all possible sets of input data; and if this program produces the correct answers for all possible sets of input data, then it is correct.

NECESSARY and SUFFICIENT CONDITIONS

If r and s are statements:

r is a sufficient condition for s means "if r then s".

r is a necessary condition for s means "if not r then not s".

r is a necessary condition for s also means "if s then r".

r is a necessary and sufficient condition for s means "r if, and only if, s".

NECESSARY and SUFFICIENT CONDITIONS

Rewrite the following statement in the form "If A then B":

1) Pia's birth on U.S soil is a sufficient condition for her to be a U.S. citizen.

Ans: If Pia was born on U.S. soil, then she is a U.S. citizen.

2) George's attaining age 35 is a necessary condition for his being president of the United States.

Ans:

Version 1: If George has not attained the age of 35, then he cannot be president of the United States.

Version 2: If George can be president of the United States, then he has attained the age of 35.

NECESSARY and SUFFICIENT CONDITIONS

Rewrite the following statement in the form "If A then B":

1) A sufficient condition for Jon's team to win the championship is that it win the rest of its games.
2) A necessary condition for this computer program to be correct is that it not produce error messages during translation.

VALID and INVALID ARGUMENTS

• Definition

An **argument** is a sequence of statements, and an **argument form** is a sequence of statement forms. All statements in an argument and all statement forms in an argument form, except for the final one, are called **premises** (or **assumptions** or **hypotheses**). The final statement or statement form is called the **conclusion.** The symbol $\therefore$, which is read "therefore," is normally placed just before the conclusion.

To say that an *argument form* is **valid** means that no matter what particular statements are substituted for the statement variables in its premises, if the resulting premises are all true, then the conclusion is also true. To say that an *argument* is **valid** means that its form is valid.

VALID and INVALID ARGUMENTS

Testing an Argument Form for Validity

1. Identify the premises and conclusion of the argument form.
2. Construct a truth table showing the truth values of all the premises and the conclusion.
3. A row of the truth table in which all the premises are true is called a **critical row**. If there is a critical row in which the conclusion is false, then it is possible for an argument of the given form to have true premises and a false conclusion, and so the argument form is invalid. If the conclusion in *every* critical row is true, then the argument form is valid.

VALID and INVALID ARGUMENTS

Determining Validity and Invalidity:

$$\begin{array}{l} p \to q \vee \sim r \\ q \to p \wedge r \\ \therefore p \to r \end{array}$$

						premises		conclusion
p	q	r	$\sim r$	$q \vee \sim r$	$p \wedge r$	$p \to q \vee \sim r$	$q \to p \wedge r$	$p \to r$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	
T	F	T	F	F	T	F	T	
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	
F	T	F	T	T	F	T	F	
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

This row shows that an argument of this form can have true premises and a false conclusion. Hence this form of argument is invalid.

VALID and INVALID ARGUMENTS

Use truth tables to determine whether the argument forms in 6–11 are valid.

6. $p \to q$
$q \to p$
$\therefore p \vee q$

7. p
$p \to q$
$\sim q \vee r$
$\therefore r$

8. $p \vee q$
$p \to \sim q$
$p \to r$
$\therefore r$

9. $p \wedge q \to \sim r$
$p \vee \sim q$
$\sim q \to p$
$\therefore \sim r$

10. $p \to r$
$q \to r$
$\therefore p \vee q \to r$

11. $p \to q \vee r$
$\sim q \vee \sim r$
$\therefore \sim p \vee \sim r$

MODUS PONENS

An argument form consisting of two premises and a conclusion is called a *syllogism*.

The first and second premises are called the *major premise* and *minor premise*, respectively.

The most famous form of syllogism in logic is called *modus ponens*. It has the following form:

$$\begin{array}{c} \text{If } p \text{ then } q. \\ p \\ \therefore q \end{array}$$

Example:

If the sum of the digits of 371,487 is divisible by 3, then 371,487 is divisible by 3.

The sum of the digits of 371,487 is divisible by 3.

∴ 371,487 is divisible by 3.

MODUS PONENS

The term *modus ponens* is Latin meaning "method of affirming".

p	q	$p \to q$	p	q	
		premises		conclusion	
T	T	T	T	T	← critical row
T	F	F	T		
F	T	T	F		
F	F	T	F		

The first row is the only one in which both premises are true, and the conclusion in that row is also true. Hence the argument form is valid.

MODUS TOLLENS

Now consider another valid argument form called *modus tollens*. It has the following form:

$$\text{If } p \text{ then } q.$$
$$\sim q$$
$$\therefore \sim p$$

The term *modus tollens* is Latin meaning "method of denying".

Example:

If Zeus is human, then Zeus is mortal.

Zeus is not mortal.

∴ Zeus is not human.

MODUS PONENS and MODUS TOLLENS

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Recognizing Modus Ponens and Modus Tollens:

Use modus ponens or modus tollens to fill in the blanks of the following arguments so that they become valid inferences.

a) If there are more pigeons than there are pigeonholes, then at least two pigeons roost in the same hole.
   There are more pigeons than there are pigeonholes.
   ∴ ______________________________.

b) If 870,232 is divisible by 6, then it is divisible by 3.
   870,232 is not divisible by 3.
   ∴ ______________________________.

MODUS PONENS and MODUS TOLLENS

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Use modus ponens or modus tollens to fill in the blanks in the arguments of 1–5 so as to produce valid inferences.

1. If $\sqrt{2}$ is rational, then $\sqrt{2} = a/b$ for some integers a and b.
   It is not true that $\sqrt{2} = a/b$ for some integers a and b.
   ∴ ______________________________.

2. If $1 - 0.99999\ldots$ is less than every positive real number, then it equals zero.
   ______________________________.
   ∴ The number $1 - 0.99999\ldots$ equals zero.

3. If logic is easy, then I am a monkey's uncle.
   I am not a monkey's uncle.
   ∴ ______________________________.

4. If this figure is a quadrilateral, then the sum of its interior angles is 360°.
   The sum of the interior angles of this figure is not 360°.
   ∴ ______________________________.

5. If they were unsure of the address, then they would have telephoned.
   ______________________________.
   ∴ They were sure of the address.

RULES OF INFERENCE

A *rule of inference* is a form of argument that is valid. Thus modus ponens and modus tollens are both rules of inference. The following are additional examples of rules of inference that are frequently used in deductive reasoning.

- Generalization
- Specialization
- Conjunction
- Elimination
- Transitivity
- Proof by Division into Cases
- Contradiction Rule

GENERALIZATION

It has the following form:

a) p b) q

$\therefore p \vee q$ $\therefore p \vee q$

These argument forms are used for making generalizations.

According to the first, if p is true, then, more generally, "p or q" is true for *any* other statement q.

Example:

Anton is a junior.

∴ (more generally) Anton is a junior or Anton is a senior.

SPECIALIZATION

It has the following form:

a) $p \wedge q$
$\therefore p$

b) $p \wedge q$
$\therefore q$

These argument forms are used for specializing.

Example:

Ana knows numerical analysis and Ana knows graph algorithms.
∴ (in particular) Ana knows graph algorithms.

CONJUNCTION

It has the following form:

p
q
$\therefore p \wedge q$

Example:

Ana knows numerical analysis.
Ana knows graph algorithms.
∴ Ana knows numerical analysis and Ana knows graph algorithms.

ELIMINATION

It has the following form:

a) $p \vee q$ b) $p \vee q$

$\sim p$ $\sim q$

$\therefore q$ $\therefore p$

These argument forms say that when you have only two possibilities and you can rule one out, the other must be the case.

Example:

$x - 3 = 0$ or $x + 2 = 0$.

If you also know that x is not negative, then $x = -2$, so

$x + 2 = 0$.

By elimination, you can then conclude that

$\therefore x - 3 = 0$.

TRANSITIVITY

It has the following form:

a) $p \rightarrow q$

$q \rightarrow r$

$\therefore p \rightarrow r$

The fact that one statement implies a second and the second implies a third, you can conclude that the first statement implies the third.

Example:

If 18,486 is divisible by 18, then 18,486 is divisible by 9.

If 18,486 is divisible by 9, then the sum of the digits of 18,486 is divisible by 9.

$\therefore$ If 18,486 is divisible by 18, then the sum of the digits of 18,486 is divisible by 9.

PROOF BY DIVISION INTO CASES

It has the following form:

a) $p \vee q$

$p \longrightarrow r$

$q \longrightarrow r$

$\therefore r$

It often happens that you know one thing or another is true. If you can show that in either case a certain conclusion follows, then this conclusion must also be true.

Example:

x is positive or x is negative.

If x is positive, then $x^2 > 0$.

If x is negative, then $x^2 > 0$.

$\therefore x^2 > 0$.

CONTRADICTION RULE

It has the following form:

a) $\sim p \longrightarrow c$

$\therefore p$

If you can show that the supposition that statement p is false leads logically to a contradiction, then you can conclude that p is true.

			premises	conclusion
p	$\sim p$	c	$\sim p \to$ c	p
T	F	F	T	T
F	T	F	F	

There is only one critical row in which the premise is true, and in this row the conclusion is also true. Hence this form of argument is valid.

SUMMARY

Table 2.3.1 Valid Argument Forms

Modus Ponens	$p \to q$ p $\therefore q$	**Elimination**	a. $p \vee q$ $\sim q$ $\therefore p$ b. $p \vee q$ $\sim p$ $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$	**Transitivity**	$p \to q$ $q \to r$ $\therefore p \to r$
Generalization	a. p $\therefore p \vee q$ b. q $\therefore p \vee q$	**Proof by Division into Cases**	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$
Specialization	a. $p \wedge q$ $\therefore p$ b. $p \wedge q$ $\therefore q$		
Conjunction	p q $\therefore p \wedge q$	**Contradiction Rule**	$\sim p \to c$ $\therefore p$

PROBLEMS

24. If Jules solved this problem correctly, then Jules obtained the answer 2.
Jules obtained the answer 2.
∴ Jules solved this problem correctly.

25. This real number is rational or it is irrational.
This real number is not rational.
∴ This real number is irrational.

26. If I go to the movies, I won't finish my homework. If I don't finish my homework, I won't do well on the exam tomorrow.
∴ If I go to the movies, I won't do well on the exam tomorrow.

31. Sandra knows Java and Sandra knows C++.
∴ Sandra knows C++.

32. If I get a Christmas bonus, I'll buy a stereo.
If I sell my motorcycle, I'll buy a stereo.
∴ If I get a Christmas bonus or I sell my motorcycle, then I'll buy a stereo.

28. If there are as many rational numbers as there are irrational numbers, then the set of all irrational numbers is infinite.
The set of all irrational numbers is infinite.
∴ There are as many rational numbers as there are irrational numbers.

29. If at least one of these two numbers is divisible by 6, then the product of these two numbers is divisible by 6.
Neither of these two numbers is divisible by 6.
∴ The product of these two numbers is not divisible by 6.

30. If this computer program is correct, then it produces the correct output when run with the test data my teacher gave me.
This computer program produces the correct output when run with the test data my teacher gave me.
∴ This computer program is correct.

PROBLEMS

Application: A More Complex Deduction

You are about to leave for school in the morning and discover that you don't have your glasses. You know the following statements are true:

a. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

b. If my glasses are on the kitchen table, then I saw them at breakfast.

c. I did not see my glasses at breakfast.

d. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.

e. If I was reading the newspaper in the living room then my glasses are on the coffee table.

Where are the glasses?

PROBLEMS

Solution Let RK = I was reading the newspaper in the kitchen.

GK = My glasses are on the kitchen table.

SB = I saw my glasses at breakfast.

RL = I was reading the newspaper in the living room.

GC = My glasses are on the coffee table.

Here is a sequence of steps you might use to reach the answer, together with the rules of inference that allow you to draw the conclusion of each step:

1. $RK \to GK$ by (a)
   $GK \to SB$ by (d)
   $\therefore RK \to SB$ by transitivity
2. $RK \to SB$ by the conclusion of (1)
   $\sim SB$ by (c)
   $\therefore \sim RK$ by modus tollens
3. $RL \vee RK$ by (d)
   $\sim RK$ by the conclusion of (2)
   $\therefore RL$ by elimination
4. $RL \to GC$ by (e)
   RL by the conclusion of (3)
   $\therefore GC$ by modus ponens

Thus the glasses are on the coffee table.

PROBLEMS

36. Given the following information about a computer program, find the mistake in the program.
    a. There is an undeclared variable or there is a syntax error in the first five lines.
    b. If there is a syntax error in the first five lines, then there is a missing semicolon or a variable name is misspelled.
    c. There is not a missing semicolon.
    d. There is not a misspelled variable name.

PROBLEMS

37. In the back of an old cupboard you discover a note signed by a pirate famous for his bizarre sense of humor and love of logical puzzles. In the note he wrote that he had hidden treasure somewhere on the property. He listed five true statements (a–e below) and challenged the reader to use them to figure out the location of the treasure.
    a. If this house is next to a lake, then the treasure is not in the kitchen.
    b. If the tree in the front yard is an elm, then the treasure is in the kitchen.
    c. This house is next to a lake.
    d. The tree in the front yard is an elm or the treasure is buried under the flagpole.
    e. If the tree in the back yard is an oak, then the treasure is in the garage.

    Where is the treasure hidden?

PROBLEMS

41. a. $\sim p \vee q \rightarrow r$
    b. $s \vee \sim q$
    c. $\sim t$
    d. $p \rightarrow t$
    e. $\sim p \wedge r \rightarrow \sim s$
    f. $\therefore \sim q$

42. a. $p \vee q$
    b. $q \rightarrow r$
    c. $p \wedge s \rightarrow t$
    d. $\sim r$
    e. $\sim q \rightarrow u \wedge s$
    f. $\therefore t$

43. a. $\sim p \rightarrow r \wedge \sim s$
    b. $t \rightarrow s$
    c. $u \rightarrow \sim p$
    d. $\sim w$
    e. $u \vee w$
    f. $\therefore \sim t$

44. a. $p \rightarrow q$
    b. $r \vee s$
    c. $\sim s \rightarrow \sim t$
    d. $\sim q \vee s$
    e. $\sim s$
    f. $\sim p \wedge r \rightarrow u$
    g. $w \vee t$
    h. $\therefore u \wedge w$

PROBLEMS

39. The famous detective Percule Hoirot was called in to solve a baffling murder mystery. He determined the following facts:
    a. Lord Hazelton, the murdered man, was killed by a blow on the head with a brass candlestick.
    b. Either Lady Hazelton or a maid, Sara, was in the dining room at the time of the murder.
    c. If the cook was in the kitchen at the time of the murder, then the butler killed Lord Hazelton with a fatal dose of strychnine.
    d. If Lady Hazelton was in the dining room at the time of the murder, then the chauffeur killed Lord Hazelton.
    e. If the cook was not in the kitchen at the time of the murder, then Sara was not in the dining room when the murder was committed.
    f. If Sara was in the dining room at the time the murder was committed, then the wine steward killed Lord Hazelton.

    Is it possible for the detective to deduce the identity of the murderer from these facts? If so, who did murder Lord Hazelton? (Assume there was only one cause of death.)

KNIGHT & KNAVES

A says: B is a knight.

B says: A and I are of opposite type.

What are A and B?

Two natives A and B address you as follows:
A says: Both of us are knights.
B says: A is a knave.
What are A and B?