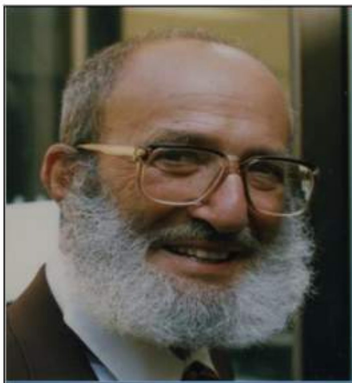

The only way to learn mathematics is to do mathematics.
— Paul Halmos —

SET THEORY

I hear and I forget.
I see and I remember.
I do and I understand.
— Confucious —

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OPERATIONS ON SETS

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• Definition

Let A and B be subsets of a universal set U.

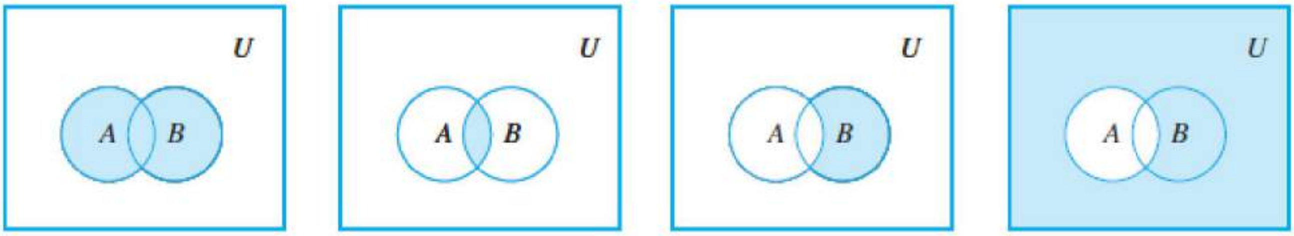
1. The **union** of A and B, denoted $A \cup B$, is the set of all elements that are in at least one of A or B.
2. The **intersection** of A and B, denoted $A \cap B$, is the set of all elements that are common to both A and B.
3. The **difference** of B minus A (or **relative complement** of A in B), denoted $B - A$, is the set of all elements that are in B and not A.
4. The **complement** of A, denoted A^c, is the set of all elements in U that are not in A.

Symbolically:

$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\},$$
$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\},$$
$$B - A = \{x \in U \mid x \in B \text{ and } x \notin A\},$$
$$A^c = \{x \in U \mid x \notin A\}.$$

OPERATIONS ON SETS

Venn diagram representations for union, intersection, difference, and complement are shown in Figure 6.1.4.



Shaded region represents $A \cup B$. | Shaded region represents $A \cap B$. | Shaded region represents $B - A$. | Shaded region represents A^c.

Figure 6.1.4

OPERATIONS ON SETS

Definition: Given sets A and B, the **symmetric difference of A and B**, denoted $A \triangle B$, is

$$A \triangle B = (A - B) \cup (B - A).$$

46. Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, and $C = \{5, 6, 7, 8\}$. Find each of the following sets:

a. $A \triangle B$ b. $B \triangle C$

c. $A \triangle C$ d. $(A \triangle B) \triangle C$

OPERATIONS ON SETS

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Unions, Intersections, Differences, and Complements

Let the universal set be the set $U = \{a, b, c, d, e, f, g\}$ and let $A = \{a, c, e, g\}$ and $B = \{d, e, f, g\}$. Find $A \cup B$, $A \cap B$, $B - A$, and A^c.

Solution

$$A \cup B = \{a, c, d, e, f, g\} \qquad A \cap B = \{e, g\}$$
$$B - A = \{d, f\} \qquad A^c = \{b, d, f\}$$

10. Let $A = \{1, 3, 5, 7, 9\}$, $B = \{3, 6, 9\}$, and $C = \{2, 4, 6, 8\}$. Find each of the following:

a. $A \cup B$ b. $A \cap B$ c. $A \cup C$ d. $A \cap C$

e. $A - B$ f. $B - A$ g. $B \cup C$ h. $B \cap C$

INTERVALS

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• Notation

Given real numbers a and b with $a \leq b$:

$$(a, b) = \{x \in \mathbf{R} \mid a < x < b\} \qquad [a,\ b] = \{x \in \mathbf{R} \mid a \leq x \leq b\}$$
$$(a, b] = \{x \in \mathbf{R} \mid a < x \leq b\} \qquad [a,\ b) = \{x \in \mathbf{R} \mid a \leq x < b\}.$$

The symbols ∞ and $-\infty$ are used to indicate intervals that are unbounded either on the right or on the left:

$$(a, \infty) = \{x \in \mathbf{R} \mid x > a\} \qquad [a, \infty) = \{x \in \mathbf{R} \mid x \geq a\}$$
$$(-\infty, b) = \{x \in \mathbf{R} \mid x < b\} \qquad [-\infty, b) = \{x \in \mathbf{R} \mid x \leq b\}.$$

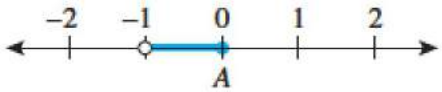
OPERATIONS ON SETS

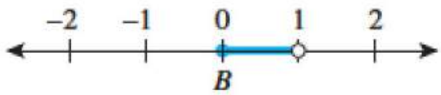
An Example with Intervals

Let the universal set be the set $\mathbf{R}$ of all real numbers and let

$$A = (-1,\ 0] = \{x \in \mathbf{R} \mid -1 < x \leq 0\} \text{ and } B = [0,\ 1) = \{x \in \mathbf{R} \mid 0 \leq x < 1\}.$$

These sets are shown on the number lines below.

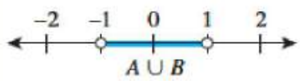




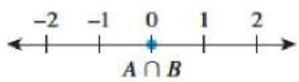
Find $A \cup B$, $A \cap B$, $B - A$, and A^c.

OPERATIONS ON SETS

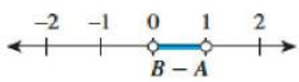
Solution



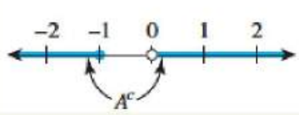
$$A \cup B = \{x \in \mathbf{R} \mid x \in (-1,\ 0] \text{ or } x \in [0,\ 1)\} = \{x \in \mathbf{R} \mid x \in (-1,\ 1)\} = (-1,\ 1).$$



$$A \cap B = \{x \in \mathbf{R} \mid x \in (-1,\ 0] \text{ and } x \in [0,\ 1)\} = \{0\}.$$



$$B - A = \{x \in \mathbf{R} \mid x \in [0,\ 1) \text{ and } x \notin (-1,\ 0]\} = \{x \in \mathbf{R} \mid 0 < x < 1\} = (0, 1)$$



$$\begin{aligned} A^c &= \{x \in \mathbf{R} \mid \text{it is not the case that } x \in (-1,\ 0]\} \\ &= \{x \in \mathbf{R} \mid \text{it is not the case that } (-1 < x \text{ and } x \leq 0)\} && \text{by definition of the double inequality} \\ &= \{x \in \mathbf{R} \mid x \leq -1 \text{ or } x > 0\} = (-\infty,\ -1] \cup (0,\ \infty) && \text{by De Morgan's law} \end{aligned}$$

OPERATIONS ON SETS

11. Let the universal set be the set $\mathbf{R}$ of all real numbers and let $A = \{x \in \mathbf{R} \mid 0 < x \le 2\}$, $B = \{x \in \mathbf{R} \mid 1 \le x < 4\}$, and $C = \{x \in \mathbf{R} \mid 3 \le x < 9\}$. Find each of the following:

a. $A \cup B$ b. $A \cap B$ c. A^c d. $A \cup C$
e. $A \cap C$ f. B^c g. $A^c \cap B^c$
h. $A^c \cup B^c$ i. $(A \cap B)^c$ j. $(A \cup B)^c$

OPERATIONS ON SETS

12. Let the universal set be the set $\mathbf{R}$ of all real numbers and let $A = \{x \in \mathbf{R} \mid -3 \le x \le 0\}$, $B = \{x \in \mathbf{R} \mid -1 < x < 2\}$, and $C = \{x \in \mathbf{R} \mid 6 < x \le 8\}$. Find each of the following:

a. $A \cup B$ b. $A \cap B$ c. A^c d. $A \cup C$
e. $A \cap C$ f. B^c g. $A^c \cap B^c$
h. $A^c \cup B^c$ i. $(A \cap B)^c$ j. $(A \cup B)^c$

OPERATIONS ON SETS

13. Indicate which of the following relationships are true and which are false:

a. $\mathbf{Z}^+ \subseteq \mathbf{Q}$

b. $\mathbf{R}^- \subseteq \mathbf{Q}$

c. $\mathbf{Q} \subseteq \mathbf{Z}$

d. $\mathbf{Z}^- \cup \mathbf{Z}^+ = \mathbf{Z}$

e. $\mathbf{Z}^- \cap \mathbf{Z}^+ = \emptyset$

f. $\mathbf{Q} \cap \mathbf{R} = \mathbf{Q}$

g. $\mathbf{Q} \cup \mathbf{Z} = \mathbf{Q}$

h. $\mathbf{Z}^+ \cap \mathbf{R} = \mathbf{Z}^+$

i. $\mathbf{Z} \cup \mathbf{Q} = \mathbf{Z}$

DISJOINT SETS

- **Definition**

Two sets are called **disjoint** if, and only if, they have no elements in common. Symbolically:

$$A \text{ and } B \text{ are disjoint} \quad \Leftrightarrow \quad A \cap B = \emptyset.$$

Disjoint Sets

Let $A = \{1, 3, 5\}$ and $B = \{2, 4, 6\}$. Are A and B disjoint?

Solution Yes. By inspection A and B have no elements in common, or, in other words, $\{1, 3, 5\} \cap \{2, 4, 6\} = \emptyset$. ■

DISJOINT SETS

• Definition

Sets $A_1, A_2, A_3 \ldots$ are **mutually disjoint** (or **pairwise disjoint** or **nonoverlapping**) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common. More precisely, for all $i, j = 1, 2, 3, \ldots$

$$A_i \cap A_j = \emptyset \quad \text{whenever } i \neq j.$$

Mutually Disjoint Sets

a. Let $A_1 = \{3, 5\}$, $A_2 = \{1, 4, 6\}$, and $A_3 = \{2\}$. Are A_1, A_2, and A_3 mutually disjoint?

b. Let $B_1 = \{2, 4, 6\}$, $B_2 = \{3, 7\}$, and $B_3 = \{4, 5\}$. Are B_1, B_2, and B_3 mutually disjoint?

PARTITION

• Definition

A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3 \ldots\}$ is a **partition** of a set A if, and only if,

1. A is the union of all the A_i
2. The sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.

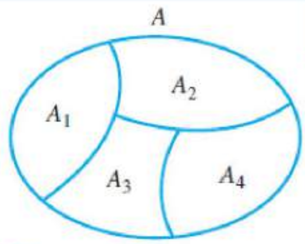


Figure 6.1.5 A Partition of a Set

PARTITION

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Partitions of Sets

a. Let $A = \{1, 2, 3, 4, 5, 6\}$, $A_1 = \{1, 2\}$, $A_2 = \{3, 4\}$, and $A_3 = \{5, 6\}$. Is $\{A_1, A_2, A_3\}$ a partition of A?

b. Let $\mathbf{Z}$ be the set of all integers and let

$$T_0 = \{n \in \mathbf{Z} \mid n = 3k, \text{ for some integer } k\},$$
$$T_1 = \{n \in \mathbf{Z} \mid n = 3k + 1, \text{ for some integer } k\}, \text{ and}$$
$$T_2 = \{n \in \mathbf{Z} \mid n = 3k + 2, \text{ for some integer } k\}.$$

Is $\{T_0, T_1, T_2\}$ a partition of $\mathbf{Z}$?

PARTITION

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27. a. Is $\{\{a, d, e\}, \{b, c\}, \{d, f\}\}$ a partition of $\{a, b, c, d, e, f\}$?

b. Is $\{\{w, x, v\}, \{u, y, q\}, \{p, z\}\}$ a partition of $\{p, q, u, v, w, x, y, z\}$?

c. Is $\{\{5, 4\}, \{7, 2\}, \{1, 3, 4\}, \{6, 8\}\}$ a partition of $\{1, 2, 3, 4, 5, 6, 7, 8\}$?

d. Is $\{\{3, 7, 8\}, \{2, 9\}, \{1, 4, 5\}\}$ a partition of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$?

e. Is $\{\{1, 5\}, \{4, 7\}, \{2, 8, 6, 3\}\}$ a partition of $\{1, 2, 3, 4, 5, 6, 7, 8\}$?

POWER SETS

- **Definition**

Given a set A, the **power set** of A, denoted $\mathscr{P}(A)$, is the set of all subsets of A.

Power Set of a Set

Find the power set of the set $\{x, y\}$. That is, find $\mathscr{P}(\{x, y\})$.

$$\mathscr{P}(\{x, y\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}.$$

POWER SETS

31. Suppose $A = \{1, 2\}$ and $B = \{2, 3\}$. Find each of the following:
    a. $\mathscr{P}(A \cap B)$ b. $\mathscr{P}(A)$
    c. $\mathscr{P}(A \cup B)$ d. $\mathscr{P}(A \times B)$

33. a. Find $\mathscr{P}(\emptyset)$. b. Find $\mathscr{P}(\mathscr{P}(\emptyset))$.
    c. Find $\mathscr{P}(\mathscr{P}(\mathscr{P}(\emptyset)))$.

PROPERTIES OF SETS

Theorem 6.2.1 Some Subset Relations

1. *Inclusion of Intersection:* For all sets A and B,

$$\text{(a) } A \cap B \subseteq A \quad \text{and} \quad \text{(b) } A \cap B \subseteq B.$$

2. *Inclusion in Union:* For all sets A and B,

$$\text{(a) } A \subseteq A \cup B \quad \text{and} \quad \text{(b) } B \subseteq A \cup B.$$

3. *Transitive Property of Subsets:* For all sets A, B, and C,

$$\text{if } A \subseteq B \text{ and } B \subseteq C, \text{ then } A \subseteq C.$$

PROPERTIES OF SETS

Theorem 6.2.2 Set Identities

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws:* For all sets A and B,

$$\text{(a) } A \cup B = B \cup A \quad \text{and} \quad \text{(b) } A \cap B = B \cap A.$$

2. *Associative Laws:* For all sets A, B, and C,

$$\text{(a) } (A \cup B) \cup C = A \cup (B \cup C) \quad \text{and}$$
$$\text{(b) } (A \cap B) \cap C = A \cap (B \cap C).$$

3. *Distributive Laws:* For all sets, A, B, and C,

$$\text{(a) } A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \quad \text{and}$$
$$\text{(b) } A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

4. *Identity Laws:* For all sets A,

$$\text{(a) } A \cup \emptyset = A \quad \text{and} \quad \text{(b) } A \cap U = A.$$

5. *Complement Laws:*

$$\text{(a) } A \cup A^c = U \quad \text{and} \quad \text{(b) } A \cap A^c = \emptyset.$$

PROPERTIES OF SETS

6. *Double Complement Law:* For all sets A,

$$(A^c)^c = A.$$

7. *Idempotent Laws:* For all sets A,

$$\text{(a) } A \cup A = A \quad \text{and} \quad \text{(b) } A \cap A = A.$$

8. *Universal Bound Laws:* For all sets A,

$$\text{(a) } A \cup U = U \quad \text{and} \quad \text{(b) } A \cap \emptyset = \emptyset.$$

9. *De Morgan's Laws:* For all sets A and B,

$$\text{(a) } (A \cup B)^c = A^c \cap B^c \quad \text{and} \quad \text{(b) } (A \cap B)^c = A^c \cup B^c.$$

10. *Absorption Laws:* For all sets A and B,

$$\text{(a) } A \cup (A \cap B) = A \quad \text{and} \quad \text{(b) } A \cap (A \cup B) = A.$$

11. *Complements of U and $\emptyset$:*

$$\text{(a) } U^c = \emptyset \quad \text{and} \quad \text{(b) } \emptyset^c = U.$$

12. *Set Difference Law:* For all sets A and B,

$$A - B = A \cap B^c.$$