The only way to learn mathematics is to do mathematics.
— Paul Halmos —

FUNCTIONS

I hear and I forget.
I see and I remember.
I do and I understand.
— Confucious —

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BASIC TERMINOLOGIES

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• Definition

A **function** f **from a set** X **to a set** Y, denoted $f\colon X \to Y$, is a relation from X, the **domain**, to Y, the **co-domain**, that satisfies two properties: (1) every element in X is related to some element in Y, and (2) no element in X is related to more than one element in Y. Thus, given any element x in X, there is a unique element in Y that is related to x by f. If we call this element y, then we say that "f sends x to y" or "f maps x to y" and write $x \xrightarrow{f} y$ or $f: x \to y$. The unique element to which f sends x is denoted

$f(x)$	and is called	f **of** x, or
		the output of f **for the input** x, or
		the value of f **at** x, or
		the image of x **under** f.

The set of all values of f taken together is called the *range of f* or the *image of X under f*. Symbolically,

$$\textbf{range of } f = \textbf{image of } X \textbf{ under } f = \{y \in Y \mid y = f(x), \text{ for some } x \text{ in } X\}.$$

Given an element y in Y, there may exist elements in X with y as their image. If $f(x) = y$, then x is called **a preimage of** y or **an inverse image of** y. The set of all inverse images of y is called *the inverse image of* y. Symbolically,

$$\textbf{the inverse image of } y = \{x \in X \mid f(x) = y\}.$$

ARROW DIAGRAM

This arrow diagram does define a function because

1. Every element of X has an arrow coming out of it.
2. No element of X has two arrows coming out of it that point to two different elements of Y.

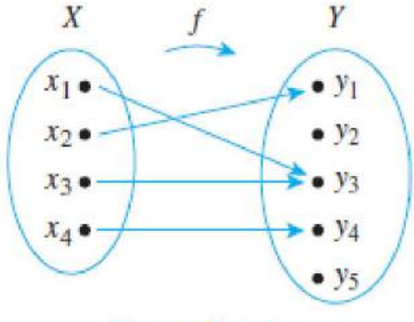
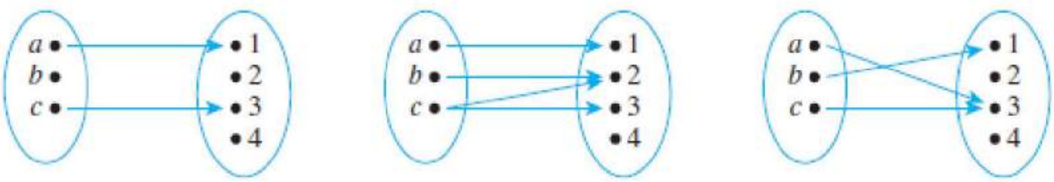


Figure 7.1.1

ARROW DIAGRAM

Functions and Nonfunctions

Which of the arrow diagrams in Figure 7.1.2 define functions from $X = \{a, b, c\}$ to $Y = \{1, 2, 3, 4\}$?



(a) (b) (c)

ARROW DIAGRAM

A Function Defined by an Arrow Diagram

Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. Define a function f from X to Y by the arrow diagram in Figure 7.1.3.

a. Write the domain and co-domain of f.

b. Find $f(a)$, $f(b)$, and $f(c)$.

c. What is the range of f?

d. Is c an inverse image of 2? Is b an inverse image of 3?

e. Find the inverse images of 2, 4, and 1.

f. Represent f as a set of ordered pairs.

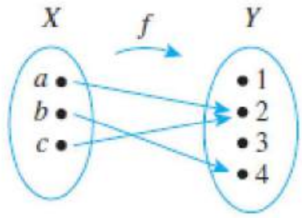
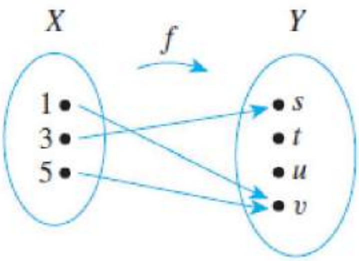


Figure 7.1.1

ARROW DIAGRAM

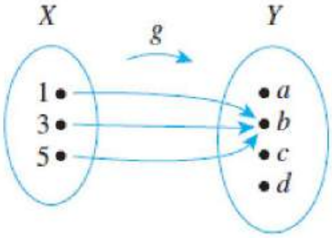
1. Let $X = \{1, 3, 5\}$ and $Y = \{s, t, u, v\}$. Define $f \colon X \to Y$ by the following arrow diagram.



a. Write the domain of f and the co-domain of f.
b. Find $f(1)$, $f(3)$, and $f(5)$.
c. What is the range of f?
d. Is 3 an inverse image of s? Is 1 an inverse image of u?
e. What is the inverse image of s? of u? of v?
f. Represent f as a set of ordered pairs.

ARROW DIAGRAM

2. Let $X = \{1, 3, 5\}$ and $Y = \{a, b, c, d\}$. Define $g: X \to Y$ by the following arrow diagram.



a. Write the domain of g and the co-domain of g.
b. Find $g(1)$, $g(3)$, and $g(5)$.
c. What is the range of g?
d. Is 3 an inverse image of a? Is 1 an inverse image of b?
e. What is the inverse image of b? of c?
f. Represent g as a set of ordered pairs.

SPECIAL FUNCTIONS

8. Let $J_5 = \{0, 1, 2, 3, 4\}$, and define a function $F: J_5 \to J_5$ as follows: For each $x \in J_5$, $F(x) = (x^3 + 2x + 4)\ mod\ 5$.

Find the following:

a. $F(0)$ b. $F(1)$ c. $F(2)$ d. $F(3)$ e. $F(4)$

9. Define a function $S: \mathbf{Z}^+ \to \mathbf{Z}^+$ as follows: For each positive integer n,

$$S(n) = \text{the sum of the positive divisors of } n.$$

Find the following:

a. $S(1)$ b. $S(15)$ c. $S(17)$
d. $S(5)$ e. $S(18)$ f. $S(21)$

SPECIAL FUNCTIONS

Define functions $M: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $R: \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ as follows:

$$M(a, b) = ab \quad \text{and} \quad R(a, b) = (-a, b).$$

Find the following:

a. $M(-1, -1)$ b. $M\left(\frac{1}{2}, \frac{1}{2}\right)$ c. $M(\sqrt{2}, \sqrt{2})$

d. $R(2, 5)$ e. $R(-2, 5)$ f. $R(3, -4)$

SPECIAL FUNCTIONS

10. Let D be the set of all finite subsets of positive integers. Define a function $T: \mathbf{Z}^+ \to D$ as follows: For each positive integer n, $T(n)$ = the set of positive divisors of n.

    Find the following:

    a. $T(1)$ b. $T(15)$ c. $T(17)$

    d. $T(5)$ e. $T(18)$ f. $T(21)$

11. Define $F: \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$ as follows: For all ordered pairs (a, b) of integers, $F(a, b) = (2a + 1, 3b - 2)$.

    Find the following:

    a. $F(4, 4)$ b. $F(2, 1)$ c. $F(3, 2)$ d. $F(1, 5)$

SPECIAL FUNCTIONS

12. Define G: $J_5 \times J_5 \to J_5 \times J_5$ as follows: For all $(a, b) \in J_5 \times J_5$,

$$G(a, b) = ((2a + 1) \; mod \; 5, (3b - 2) \; mod \; 5).$$

Find the following:

a. $G(4, 4)$ b. $G(2, 1)$ c. $G(3, 2)$ d. $G(1, 5)$

FUNCTION EQUALITY

Theorem 7.1.1 A Test for Function Equality

If $F: X \to Y$ and $G: X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.

FUNCTION EQUALITY

Equality of Functions

a. Let $J_3 = \{0, 1, 2\}$, and define functions f and g from J_3 to J_3 as follows: For all x in J_3,

$$f(x) = (x^2 + x + 1)\ mod\ 3 \quad \text{and} \quad g(x) = (x + 2)^2\ mod\ 3.$$

Does $f = g$?

a. Yes, the table of values shows that $f(x) = g(x)$ for all x in J_3.

x	x^2+x+1	$f(x) = (x^2+x+1)\ mod\ 3$	$(x+2)^2$	$g(x) = (x+2)^2\ mod\ 3$
0	1	$1\ mod\ 3 = 1$	4	$4\ mod\ 3 = 1$
1	3	$3\ mod\ 3 = 0$	9	$9\ mod\ 3 = 0$
2	7	$7\ mod\ 3 = 1$	16	$16\ mod\ 3 = 1$

FUNCTION EQUALITY

13. Let $J_5 = \{0, 1, 2, 3, 4\}$, and define functions $f : J_5 \to J_5$ and $g : J_5 \to J_5$ as follows: For each $x \in J_5$,

$$f(x) = (x + 4)^2\ mod\ 5 \quad \text{and} \quad g(x) = (x^2 + 3x + 1)\ mod\ 5.$$

Is $f = g$? Explain.

14. Let $J_5 = \{0, 1, 2, 3, 4\}$, and define functions $h : J_5 \to J_5$ and $k : J_5 \to J_5$ as follows: For each $x \in J_5$,

$$h(x) = (x + 3)^3\ mod\ 5 \quad \text{and} \quad k(x) = (x^3 + 4x^2 + 2x + 2)\ mod\ 5.$$

Is $h = k$? Explain.

LOGARITHMIC FUNCTIONS

• Definition Logarithms and Logarithmic Functions

Let b be a positive real number with $b \neq 1$. For each positive real number x, the **logarithm with base b of x,** written $\log_b x$, is the exponent to which b must be raised to obtain x. Symbolically,

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x.$$

The **logarithmic function with base b** is the function from $\mathbf{R}^+$ to $\mathbf{R}$ that takes each positive real number x to $\log_b x$.

LOGARITHMIC FUNCTIONS

The Logarithmic Function with Base b

Find the following:

a. $\log_3 9$ b. $\log_2\left(\frac{1}{2}\right)$ c. $\log_{10}(1)$ d. $\log_2(2^m)$ (m is any real number)

e. $2^{\log_2 m}(m > 0)$

LOGARITHMIC FUNCTIONS

17. Use the definition of logarithm to fill in the blanks below.
    a. $\log_2 8 = 3$ because ______.
    b. $\log_5 \left(\frac{1}{25}\right) = 2$ because ______.
    c. $\log_4 4 = 1$ because ______.
    d. $\log_3(3^n) = n$ because ______.
    e. $\log_4 1 = 0$ because ______.

18. Find exact values for each of the following quantities. Do not use a calculator.
    a. $\log_3 81$
    b. $\log_2 1024$
    c. $\log_3 \left(\frac{1}{27}\right)$
    d. $\log_2 1$
    e. $\log_{10} \left(\frac{1}{10}\right)$
    f. $\log_3 3$
    g. $\log_2(2^k)$

FUNCTIONS ACTING ON SETS

• Definition

If $f: X \to Y$ is a function and $A \subseteq X$ and $C \subseteq Y$, then

$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \text{ in } A\}$$

and

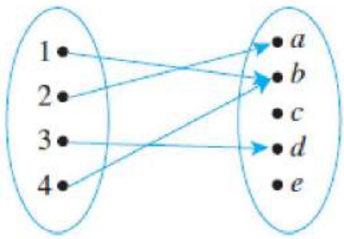
$$f^{-1}(C) = \{x \in X \mid f(x) \in C\}.$$

$f(A)$ is called the **image of** A, and $f^{-1}(C)$ is called the **inverse image of** C.

FUNCTIONS ACTING ON SETS

The Action of a Function on Subsets of a Set

Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d, e\}$, and define $F\colon\ X \to Y$ by the following arrow diagram:



Let $A = \{1, 4\}$, $C = \{a, b\}$, and $D = \{c, e\}$. Find $F(A)$, $F(X)$, $F^{-1}(C)$, and $F^{-1}(D)$.

FUNCTIONS ACTING ON SETS

38. Let $X = \{a, b, c\}$ and $Y = \{r, s, t, u, v, w\}$. Define $f\colon X \to Y$ as follows: $f(a) = v$, $f(b) = v$, and $f(c) = t$.
    a. Draw an arrow diagram for f.
    b. Let $A = \{a, b\}$, $C = \{t\}$, $D = \{u, v\}$, and $E = \{r, s\}$. Find $f(A)$, $f(X)$, $f^{-1}(C)$, $f^{-1}(D)$, $f^{-1}(E)$, and $f^{-1}(Y)$.

39. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d, e\}$. Define $g\colon\ X \to Y$ as follows: $g(1) = a$, $g(2) = a$, $g(3) = a$, and $g(4) = d$.
    a. Draw an arrow diagram for g.
    b. Let $A = \{2, 3\}$, $C = \{a\}$, and $D = \{b, c\}$. Find $g(A)$, $g(X)$, $g^{-1}(C)$, $g^{-1}(D)$, and $g^{-1}(Y)$.

ONE-TO-ONE FUNCTIONS

• Definition

Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) if, and only if, for all elements x_1 and x_2 in X,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently, $$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

Symbolically,

$$F\colon X \to Y \text{ is one-to-one} \quad \Leftrightarrow \quad \forall x_1, x_2 \in X, \text{if } F(x_1) = F(x_2) \text{ then } x_1 = x_2.$$

A function $F\colon X \to Y$ is *not* one-to-one $\Leftrightarrow$ $\exists$ elements x_1 and x_2 in X with $F(x_1) = F(x_2)$ and $x_1 \neq x_2$.

ONE-TO-ONE FUNCTIONS

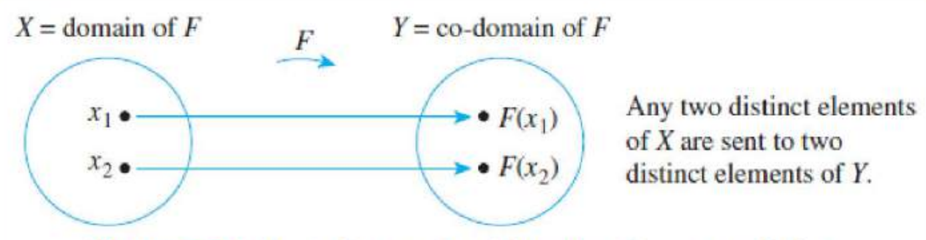


Figure 7.2.1(a) **A One-to-One Function Separates Points**

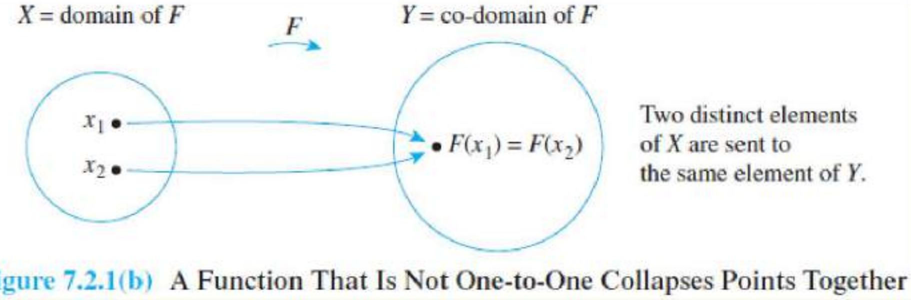


Figure 7.2.1(b) **A Function That Is Not One-to-One Collapses Points Together**

ONE-TO-ONE FUNCTIONS

a. Do either of the arrow diagrams in Figure 7.2.2 define one-to-one functions?

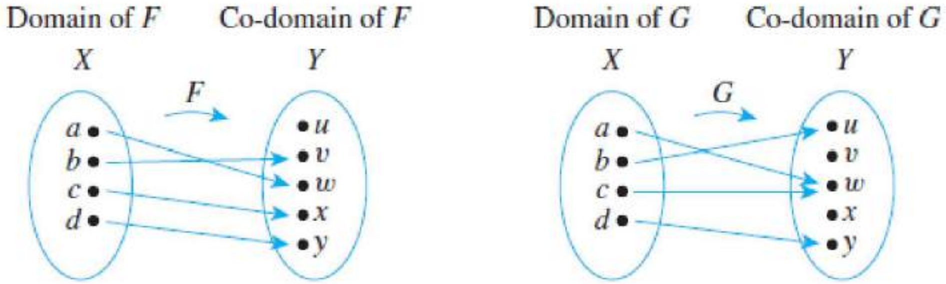


Figure 7.2.2

b. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Define $H: X \to Y$ as follows: $H(1) = c$, $H(2) = a$, and $H(3) = d$. Define $K: X \to Y$ as follows: $K(1) = d$, $K(2) = b$, and $K(3) = d$. Is either H or K one-to-one?

ONE-TO-ONE FUNCTIONS

One-to-One Functions of Infinite Sets:

Now suppose *f* is a function defined on an infinite set *X*. By definition, *f* is one-to-one if, and only if

$$\forall\, x_1, x_2 \in X, if\ f(x_1) = f(x_2)\ then\ x_1 = x_2.$$

Thus, to prove *f* is one-to-one,

Suppose x_1 *and* x_2 are elements of X such that $f(x_1) = f(x_2)$ and show that $x_1 = x_2$.

To show that *f* is *not* one-to-one,

Find elements x_1 *and* x_2 in X so that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

ONE-TO-ONE FUNCTIONS

Proving or Disproving that Functions are One-to-One:

Define $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{Z} \to \mathbf{Z}$ by the rules

$$f(x) = 4x - 1 \quad \text{for all} \quad x \in \mathbf{R}$$

and

$$g(n) = n^2 \quad \text{for all} \quad n \in \mathbf{Z}.$$

a. Is f one-to-one? Prove or give a counterexample.

b. Is g one-to-one? Prove or give a counterexample.

ONTO FUNCTIONS

• Definition

Let F be a function from a set X to a set Y. F is **onto** (or **surjective**) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.

Symbolically:

$$F: X \to Y \text{ is onto} \quad \Leftrightarrow \quad \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

$$F: X \to Y \text{ is } \textit{not} \text{ onto} \quad \Leftrightarrow \quad \exists y \text{ in } Y \text{ such that } \forall x \in X, F(x) \neq y.$$

ONTO FUNCTIONS

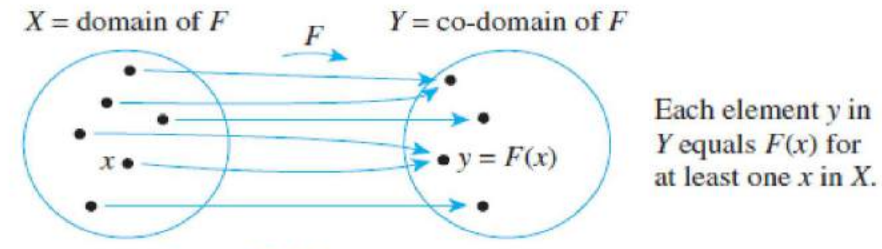


Figure 7.2.3(a) **A Function That Is Onto**

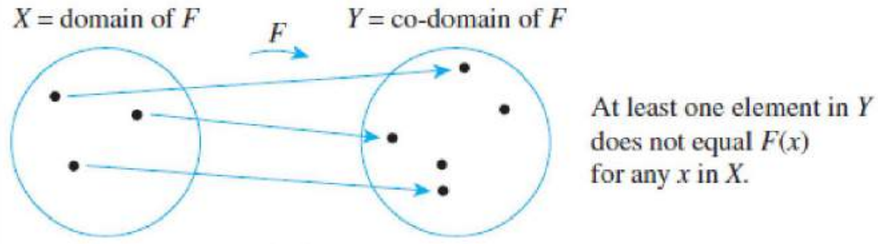


Figure 7.2.3(b) **A Function That Is Not Onto**

ONTO FUNCTIONS

Identifying Onto Functions Defined on Finite Sets

a. Do either of the arrow diagrams in Figure 7.2.4 define onto functions?

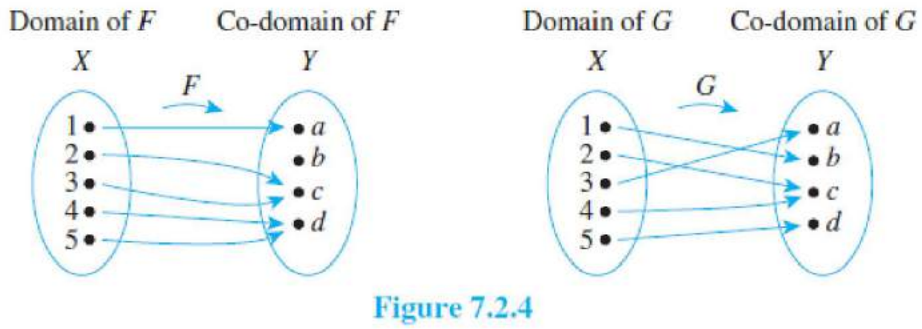


Figure 7.2.4

b. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c\}$. Define $H: X \to Y$ as follows: $H(1) = c$, $H(2) = a$, $H(3) = c$, $H(4) = b$. Define $K: X \to Y$ as follows: $K(1) = c$, $K(2) = b$, $K(3) = b$, and $K(4) = c$. Is either H or K onto?

ONTO FUNCTIONS

Onto Functions on Infinite Sets:

Now suppose F is a function from a set X to a set Y, and suppose Y is infinite. By definition, F is onto if, and only if,

$$\forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

Thus, to prove F is onto,

suppose that y is any element of Y

and show that there is an element x of X with $F(x) = y$.

To prove F is *not* onto,

find an element y of Y such that $y \neq F(x)$ for *any* x in X.

ONTO FUNCTIONS

Proving or Disproving that Functions are Onto:

Define f: $\mathbf{R} \to \mathbf{R}$ and h: $\mathbf{Z} \to \mathbf{Z}$ by the rules

$$f(x) = 4x - 1 \quad \text{for all } x \in \mathbf{R}$$

and

$$h(n) = 4n - 1 \quad \text{for all } n \in \mathbf{Z}.$$

a. Is f onto? Prove or give a counterexample.

b. Is h onto? Prove or give a counterexample.

BIJECTIVE FUNCTIONS

Let F be a function from a set X to a set Y. F is bijective function if, and only if, F is both one-to-one and onto.

Example:

The function $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by the formula

$f(x) = 4x - 1$ for all real numbers x.

INVERSE FUNCTIONS

Suppose $F: X \longrightarrow Y$ is a bijective function. Then there is a function $F^{-1}: Y \longrightarrow X$ that is defined as follows:

Given any element y in Y,

$$\mathrm{F}^{-1}(y) = \textit{that unique element } x \textit{ in } X \textit{ such that } F(x) = y$$

In other words,

$$F^{-1}(y) = x \quad \Leftrightarrow \quad y = F(x).$$

Example:

The function $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by the formula

$f(x) = 4x - 1$ for all real numbers x.

INVERSE FUNCTIONS

The function $g: \mathbb{R} \longrightarrow \mathbb{R}$ defined by the formula

$$g(x) = 2 - 3x \text{ for all real numbers } x.$$

Is g bijective function? If yes, find g^{-1}.

COMPOSITION OF FUNCTIONS

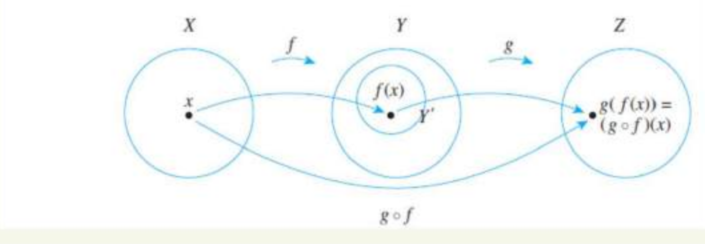
• Definition

Let $f: X \to Y'$ and $g: Y \to Z$ be functions with the property that the range of f is a subset of the domain of g. Define a new function $g \circ f: X \to Z$ as follows:

$$(g \circ f)(x) = g(f(x)) \quad \text{for all } x \in X,$$

where $g \circ f$ is read "g circle f" and $g(f(x))$ is read "g of f of x." The function $g \circ f$ is called the **composition of f and g.**

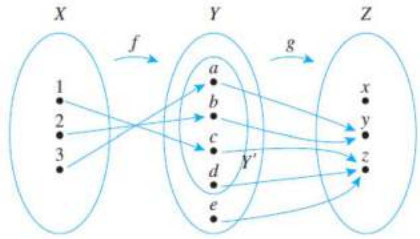
This definition is shown schematically below.



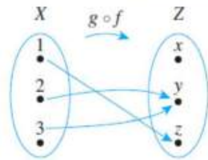
COMPOSITION OF FUNCTIONS

Composition of Functions Defined on Finite Sets

Let $X = \{1, 2, 3\}$, $Y' = \{a, b, c, d\}$, $Y = \{a, b, c, d, e\}$, and $Z = \{x, y, z\}$. Define functions $f: X \to Y'$ and $g: Y \to Z$ by the arrow diagrams below.



Draw the arrow diagram for $g \circ f$. What is the range of $g \circ f$?

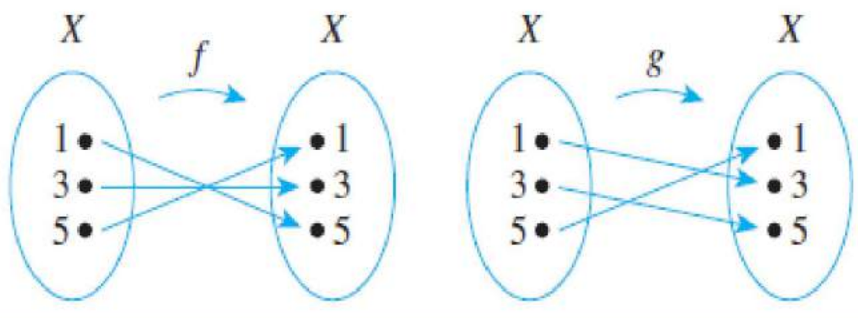


$(g \circ f)(1) = g(f(1)) = g(c) = z$
$(g \circ f)(2) = g(f(2)) = g(b) = y$
$(g \circ f)(3) = g(f(3)) = g(a) = y$

The range of $g \circ f$ is $\{y, z\}$.

COMPOSITION OF FUNCTIONS

Find gof and fog and determine whether $gof = fog$.



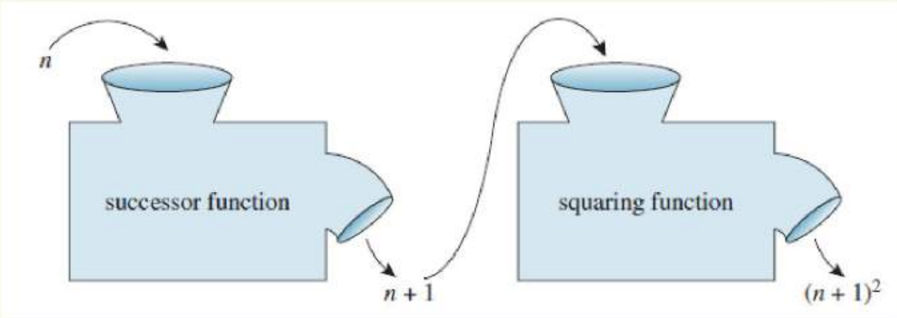
COMPOSITION OF FUNCTIONS

Composition of Functions Defined by Formula:

Let f: $\mathbf{Z} \to \mathbf{Z}$ be the successor function and let g: $\mathbf{Z} \to \mathbf{Z}$ be the squaring function. Then $f(n) = n + 1$ for all $n \in \mathbf{Z}$ and $g(n) = n^2$ for all $n \in \mathbf{Z}$.

a. Find the compositions $g \circ f$ and $f \circ g$.

b. Is $g \circ f = f \circ g$? Explain.



COMPOSITION OF FUNCTIONS

Find GoF and FoG and determine whether $GoF = FoG$.

$F(x) = x^3$ and $G(x) = x - 1$, for all real numbers x.

COMPOSITION OF FUNCTIONS

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5. Define $f: \mathbf{R} \to \mathbf{R}$ by the rule $f(x) = -x$ for all real numbers x. Find $(f \circ f)(x)$.

6. Define $F: \mathbf{Z} \to \mathbf{Z}$ and $G: \mathbf{Z} \to \mathbf{Z}$ by the rules $F(a) = 7a$ and $G(a) = a \bmod 5$ for all integers a. Find $(G \circ F)(0)$, $(G \circ F)(1)$, $(G \circ F)(2)$, $(G \circ F)(3)$, and $(G \circ F)(4)$.

7. Define $H: \mathbf{Z} \to \mathbf{Z}$ and $K: \mathbf{Z} \to \mathbf{Z}$ by the rules $H(a) = 6a$ and $K(a) = a\, mod\ 4$ for all integers a. Find $(K \circ H)(0)$, $(K \circ H)(1)$, $(K \circ H)(2)$, and $(K \circ H)(3)$.

8. Define $L: \mathbf{Z} \to \mathbf{Z}$ and $M: \mathbf{Z} \to \mathbf{Z}$ by the rules $L(a) = a^2$ and $M(a) = a\, mod\ 5$ for all integers a.
   a. Find $(L \circ M)(12)$, $(M \circ L)(12)$, $(L \circ M)(9)$, and $(M \circ L)(9)$.
   b. Is $L \circ M = M \circ L$?