Chapter - 4

Elementary Number Theory and Methods of Proof.

* 4.1 Direct Proof and Counterexample I:-

* Definitions:

An integer n is even iff n equals twice some integer.
An integer n is odd iff n equals twice some integer plus one.
Symbolically, if n is an integer then
n is even $\Leftrightarrow$ $\exists$ an integer k such that $n = 2k$.
n is odd $\Leftrightarrow$ $\exists$ an integer k such that $n = 2k+1$.

* Use the definitions of even and odd to justify your answer to the following questions.

a) Is 0 even?
Ans. Yes. As $0 = 2 \times 0$.

b) Is -301 is odd?
Ans. Yes. As $-301 = 2 \times (-151) + 1$

c) If a and b are integers, is $6a^2b$ even?
Ans. Yes. As $6a^2b = 2(3a^2b) = 2k$ where $k = 3a^2b \in \mathbb{Z}$.

d) If a and b are integers, is $10a+8b+1$ odd?
Ans. Yes. As $10a+8b+1 = 2(5a+4b)+1 = 2k+1$ where $k = 5a+4b \in \mathbb{Z}$.

* Note: Every integer is either even or odd.

* Definition:

An integer n is prime iff $n>1$ and for all positive integers r and s, if $n=rs$ then either r or s equals n.
An integer n is composite iff $n>1$ and $n=rs$ for some integers r and s with $1<r<n$ and $1<s<n$.

Symbolically:

n is prime $\Leftrightarrow \forall$ positive integers r and s, if $n=rs$ then either $r=1$ and $s=n$ or $r=n$ and $s=1$.

n is composite $\Leftrightarrow \exists$ positive integers r and s, such that $n=rs$ and $1<r<n$ and $1<s<n$.

* Write the first six prime numbers.

Ans. 2, 3, 5, 7, 11, 13.

* Write the first six composite numbers.

Ans. 4, 6, 8, 9, 10, 12.

→ Note: Every integer greater than 1 is either prime or composite.

* Proving Existential statements:

(ie) to prove statements of the form "$\exists x \in D$ such that $Q(x)$."

* Prove that $\exists$ an integer n that can be written in to two ways a sum of two prime numbers.

Ans. Let $n=10$. Then $n=5+5$ and $n=3+7$.

* Suppose that r and s are integers. Prove that $\exists$ an integer k such that $22r+18s=2k$.

Ans. Consider, $22r+18s = 2(11r+9s) = 2k$ where $k=11r+9s \in \mathbb{Z}$.

* Disproving Universal statements by Counterexample:

To disprove a statement of the form " $\forall x \in D$, if $P(x)$ then $Q(x)$", find a value of x in D for which the hypothesis $P(x)$ is true and the conclusion $Q(x)$ is false.
Such an x is called a counterexample.

* Disprove the statement: $\forall$ real numbers a and b, if $a^2 = b^2$ then $a = b$.

Soln: Let $a = 1$ and $b = -1$.
Then $a^2 = 1^2 = 1$ and $b^2 = (-1)^2 = 1$
$\therefore \; a^2 = b^2$.
But $a \neq b$.

* Proving Universal Statements:

(ie) to prove statements of the form, " $\forall x \in D$, if $P(x)$ then $Q(x)$".
We can prove an universal statement by two methods.

1. The Method of Exhausion
2. Generalize from the Generic Particular

In the method of Exhausion we prove the statement for each $x \in D$ and which is possible only if domain (ie) D of value x is finite. If the domain D contains infinitely many values we use method of Generalizing from the Generic Particular. In this method, we suppose x is a particular but arbitrarily chosen element of the domain and show that x satisfies the statement.

* Prove that, $\forall n \in \mathbb{Z}$, if n is even and $4 \le n \le 26$, then n can be written as a sum of two prime numbers.

Solⁿ:- $4 = 2+2$, $6 = 3+3$, $8 = 3+5$, $10 = 5+5$, $12 = 5+7$, $14 = 7+7$, $16 = 5+11$, $18 = 7+11$, $20 = 7+13$, $22 = 11+11$, $24 = 5+19$, $26 = 13+13$

Hence proved.

* Note: The method used to prove above statement is "The Method Of Exhausion".

* Method of Direct Proof by Generalizing from Generic Particular:

Step I: Express the statement to be proved in the form "$\forall x \in D$, if $P(x)$ then $Q(x)$." (if required)
(This step is often done mentally)

Step II: Start the proof by supposing x is a particular but arbitrarily chosen element of D for which the hypothesis $P(x)$ is true
(This step is often abbreviated as "Suppose $x \in D$ such that $P(x)$")

Step III: Show that the conclusion $Q(x)$ is true by using definitions, previously established results and the rules for logical inference.

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Theorem: The sum of any two even integers is even.

~~Proof~~ (ie) to prove,

$\forall\, m, n \in \mathbb{Z}$, if m and n are even then $m+n$ is even.

Proof: Suppose $m, n \in \mathbb{Z}$ such that m and n are even.

By definition of even integers,

$m = 2k_1$ for some $k_1 \in \mathbb{Z}$

$n = 2k_2$ for some $k_2 \in \mathbb{Z}$

Consider, $m+n = 2k_1 + 2k_2$

$= 2(k_1 + k_2)$

$= 2k$ where $k = k_1 + k_2 \in \mathbb{Z}$.

$\therefore$ $m+n$ is even integer.

Exercise – 4.1

* Answer the following questions with justification:

1. Assume that k is a particular integer.

a. Is -17 an odd integer?

Ans. Yes. As $-17 = 2 \times (-9) + 1$

b. Is 0 an even integer?

Ans. Yes. As $0 = 2 \times 0$.

c. Is $2k-1$ odd?

Ans. Yes. As $2k-1 = 2k-1+1-1 = 2k-2+1 = 2(k-1)+1 = 2k'+1$ where $k' = k-1 \in \mathbb{Z}$.

2. Assume that m and n are particular integers.?

a. Is $6m+8n$ even?

Ans. Yes. As $6m+8n = 2(3m+4n) = 2k$ where $k = 3m+4n \in \mathbb{Z}$

b. Is $10mn+7$ odd?

Ans. Yes. As $10mn+7 = 10mn+6+1 = 2(5mn+3)+1$
$= 2k+1$ where $k = 5mn+3 \in \mathbb{Z}$.

c. If $m>n>0$, is m^2-n^2 composite?

Ans. No. Let $m = 4$ and $n = 3$

Then $m>n>0$ and $m^2-n^2 = 16-9 = 7$ which is prime.

3. Assume that r and s are particular integers.

a. Is $4rs$ even?

Ans. Yes. As $4rs = 2(2rs) = 2k$ where $k = 2rs \in \mathbb{Z}$.

b. Is $6r+4s^2+3$ odd?

Ans. Yes. As $6r+4s^2+3 = 6r+4s^2+2+1 = 2(3r+2s^2+1)+1$
$= 2k+1$ where $k = 3r+2s^2+1 \in \mathbb{Z}$.

c. If r and s are both positive, is $r^2+2rs+s^2$ composite?

Ans. Yes. As $r^2+2rs+s^2 = (r+s)^2 = (r+s)(r+s) = mn$ —(i)
where $m = r+s$ and $n = r+s$.

Since r and s both positive, $r, s \geq 1 \Rightarrow r+s \geq 2$
$\Rightarrow m$ and $n \geq 2 > 1$ so $1 < m$ and $1 < n$ —(ii)

Also if $x>1$ then $x < x^2$
we have m and $n > 1$
$\Rightarrow m < m^2$ and $n < n^2$
$\Rightarrow m < (r+s)^2 = r^2+2rs+s^2$ and $n < (r+s)^2 = r^2+2rs+s^2$ —(iii)

By (i), (ii) and (iii),
$r^2+2rs+s^2 = mn$
and $1 < m < r^2+2rs+s^2$, $1 < n < r^2+2rs+s^2$

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* Prove the following:

4. There are integers m and n such that $m>1$ and $n>1$ and $\frac{1}{m}+\frac{1}{n}$ is an integer.

Soln:- Let $m=2$ and $n=2$.

Then $\frac{1}{m}+\frac{1}{n} = \frac{1}{2}+\frac{1}{2} = 1 \in \mathbb{Z}$.

5. There are distinct integers m and n such that ~~[illegible]~~ and $\frac{1}{m}+\frac{1}{n}$ is an integer.

Soln:- Let $m=1$ and $n=-1$.

Then $\frac{1}{m}+\frac{1}{n} = \frac{1}{1}+\frac{1}{-1} = 1-1 = 0 \in \mathbb{Z}$.

6. There are real numbers a and b such that $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$.

Soln: Let $a=0$ and $b=1$

Then $\sqrt{a+b} = \sqrt{0+1} = 1$ and

$\sqrt{a} = \sqrt{0} = 0$, $\sqrt{b} = \sqrt{1} = 1$

$\Rightarrow \sqrt{a}+\sqrt{b} = 0+1$

$\therefore \sqrt{a+b} = \sqrt{a}+\sqrt{b}$

7. There is an integer $n>5$ such that 2^n-1 is prime.

Soln:- ~~L~~ Let $n=7$

Then $2^n-1 = 2^7-1 = 128-1 = 127$ which is prime.

9. There is a perfect square that can be written as a sum two other perfect squares.

Soln:- $25 = 9+16$

10. There is an integer n such that $2n^2-5n+2$ is prime.

Sol^n: Let $n=3$

Then $2n^2-5n+2 = 2\times 9 - 5\times 3+2 = 18-15+2 = 5$.

* Disprove the statements by giving counterexample.

11. For all real numbers a and b, if $a<b$ then $a^2<b^2$.

Sol^n: Let $a=-2$ and $b=-1 \Rightarrow a^2=4$ and $b^2=1$.

Then $a<b$ and $a^2>b^2$.

12. For all integers n, if n is odd then $\frac{n-1}{2}$ is odd.

Sol^n: Let $n=5$. $\Rightarrow \frac{n-1}{2} = \frac{5-1}{2} = 2$

Then n is odd and $\frac{n-1}{2}$ is even.

13. For all integers m and n, if $2m+n$ is odd then m and n are both odd.

Sol^n: Let $m=2$ and $n=3 \Rightarrow 2m+n = 2\times 2+3 = 7$

Then $2m+n$ is odd ~~and~~ but m is even and n is odd.

even

14. Every positive integer less than 26 can be expressed as a sum of three or fewer perfect squares.

Sol^n: ~~$1=1^2$, $2=1^2+1^2$, $3=1^2+1^2+1^2$, $4=2^2$, $5=1^2+2^2$, $6=1^2+1^2+2^2$, $7=$~~

$2=1^2+1^2$, $4=2^2$, $6=1^2+1^2+2^2$, $8=2^2+2^2$, $10=1^2+3^2$

$12=2^2+2^2+2^2$, $14=1^2+2^2+3^2$, $16=4^2$, $18=3^2+3^2$, $20=2^2+4^2$

$22=3^2+3^2+2^2$, $24=2^2+2^2+4^2$.

Hence proved.

18. For each integer n with $1 \le n \le 10$, $n^2 - n + 11$ is a prime number.

Soln: $1^2 - 1 + 11 = 11$, $2^2 - 2 + 11 = 13$, $3^2 - 3 + 11 = 17$,
$4^2 - 4 + 11 = 23$, $5^2 - 5 + 11 = 31$, $6^2 - 6 + 11 = 41$, $7^2 - 7 + 11 = 53$,
$8^2 - 8 + 11 = 67$, $9^2 - 9 + 11 = 83$, $10^2 - 10 + 11 = 101$.

Hence proved.

Thm:

19. Prove that the sum of any even integer and any odd integer is odd.

(ie) to prove, $\forall m, n \in \mathbb{Z}$ if m is even and n is odd then $m+n$ is odd.

Proof:- Let $m, n \in \mathbb{Z}$ such that m is even and n is odd.

By definition of even and odd integers,

$m = 2k_1$ for some $k_1 \in \mathbb{Z}$

$n = 2k_2 + 1$ for some $k_2 \in \mathbb{Z}$.

$\therefore m+n = 2k_1 + 2k_2 + 1$

$= 2(k_1 + k_2) + 1$

$= 2k + 1$ where $k = k_1 + k_2 \in \mathbb{Z}$

$\therefore m+n$ is odd.

Hence proved.

24 Prove that negative of any even integer is even.

~~Proof~~ (ie) to prove, $\forall n \in \mathbb{Z}$, if n is even then $-n$ is even.

Proof! Let $n \in \mathbb{Z}$ such that n is even.

By definition of even,

$n = 2k$ for some $k \in \mathbb{Z}$

$\therefore -n = -2k = 2(-k) = 2k'$ where $k' = -k \in \mathbb{Z}$

$\therefore -n$ is even.

Hence proved.

Prove that,

25. The difference of any even integer minus any odd integer is odd.

(ie) to prove, $\forall\, m, n \in \mathbb{Z}$, if m is even and n is odd then $m-n$ is odd.

Proof: Let $m, n \in \mathbb{Z}$ such that m is even and n is odd.

By definition of even and odd integers,

$m = 2k_1$ for some $k_1 \in \mathbb{Z}$

$n = 2k_2 + 1$ for some $k_2 \in \mathbb{Z}$.

$\therefore\; m-n = 2k_1 - 2k_2 - 1$

$= 2k_1 - 2k_2 - 1 + 1 - 1$

$= 2k_1 - 2k_2 - 2 + 1$

$= 2(k_1 - k_2 - 1) + 1$

$= 2k + 1$ where $k = k_1 - k_2 - 1 \in \mathbb{Z}$.

$\therefore\; m-n$ is odd.

Hence proved.

Prove that,

26. The difference of any odd integer and any even integer is odd.

(ie) to prove, $\forall\, m, n \in \mathbb{Z}$, if m is odd and n is even then $m-n$ is odd.

Proof: Let $m, n \in \mathbb{Z}$ such that m is odd and n is even.

By definition of even and odd integers,

$m = 2k_1 + 1$ for some $k_1 \in \mathbb{Z}$

$n = 2k_2$ for some $k_2 \in \mathbb{Z}$

$\therefore\; m-n = 2k_1 + 1 - 2k_2$

$= 2(k_1 - k_2) + 1$

$= 2k + 1$ where $k = k_1 - k_2 \in \mathbb{Z}$.

$\therefore\; m-n$ is odd.

Hence proved.

Prove that,

27. The sum of any two odd integers is even.

(ie) to prove $\forall m, n \in \mathbb{Z}$, if m and n are odd then $m+n$ is even.

Proof: Let $m, n \in \mathbb{Z}$ such that m and n are odd.

By definition of odd integers,

$m = 2k_1+1$ for some $k_1 \in \mathbb{Z}$

$n = 2k_2+1$ for some $k_2 \in \mathbb{Z}$

$\therefore m+n = 2k_1+1+2k_2+1$

$= 2k_1+2k_2+2$

$= 2(k_1+k_2+1)$

$= 2k$ where $k = k_1+k_2+1 \in \mathbb{Z}$.

$\therefore m+n$ is even

Hence proved.

28. Prove that, $\forall$ integers n, if n is odd then n^2 is odd.

Proof: Let $n \in \mathbb{Z}$ such that n is odd.

By definition of odd integers,

$n = 2k+1$ for some $k \in \mathbb{Z}$.

$\therefore n^2 = (2k+1)^2$

$= 4k^2+4k+1$

$= 2(2k^2+2k)+1$

$= 2k'+1$ where $k' = 2k^2+2k \in \mathbb{Z}$.

29. Prove that, $\forall$ integers n, if n is odd then $3n+5$ is even.

Proof. Let $n \in \mathbb{Z}$ such that n is odd.

By definition of odd integers,

$n = 2k+1$ for some $k \in \mathbb{Z}$.

$\therefore 3n+5 = 3(2k+1)+5 = 6k+3+5 = 6k+8 = 2(3k+4)$

$= 2k'$ where $k' = 3k+4 \in \mathbb{Z}$

$\therefore 3n+5$ is even.

Hence proved.

30. Prove that, ∀ integers m, if m is even then $3m+5$ is odd

Proof: Let $m \in \mathbb{Z}$ such that m is even.

By definition of even integers,

$m = 2k$ for some $k \in \mathbb{Z}$.

$\therefore 3m+5 = 3(2k)+5 = 6k+5 = 6k+4+1 = 2(3k+2)+1$

$= 2k'+1$ where $k' = 3k+2 \in \mathbb{Z}$.

$\therefore 3m+5$ is odd.

Hence proved.

31. Prove that If k is any odd integer and m is any even integer then k^2+m^2 is odd.

Proof: Let $k, m \in \mathbb{Z}$ such that k is odd and m is even.

By definition of odd and even integers,

$k = 2t_1+1$ for some $t_1 \in \mathbb{Z}$

$m = 2t_2$ for some $t_2 \in \mathbb{Z}$.

$\therefore k^2+m^2 = (2t_1+1)^2 + (2t_2)^2$

$= 4t_1^2 + 4t_1 + 1 + 4t_2^2$

$= 2(2t_1^2 + 2t_1 + 2t_2^2) + 1$

$= 2t+1$ where $t = 2t_1^2 + 2t_1 + 2t_2^2$

$\therefore k^2+m^2$ is odd.

Hence proved.

32. Prove that, if a is any odd integer and b is any even integer then $2a+3b$ is even.

Proof: Let $a, b \in \mathbb{Z}$ such that a is odd and b is even.

By definition of odd and even integers,

$a = 2k_1+1$ for some $k_1 \in \mathbb{Z}$

$b = 2k_2$ for some $k_2 \in \mathbb{Z}$.

$\therefore 2a+3b = 2(2k_1+1) + 3(2k_2) = 4k_1+2+6k_2 = 2(2k_1+3k_2+1)$

$= 2k$ where $k = 2k_1+3k_2+1 \in \mathbb{Z}$

$\therefore 2a+3b$ is even. Hence proved.

33. Prove that, if n is any even integer, then $(-1)^n = 1$.

Proof: Let $n \in \mathbb{Z}$ such that n is even.

By definition of even integers,

$n = 2k$ for some $k \in \mathbb{Z}$.

$\therefore (-1)^n = (-1)^{2k} = ((-1)^2)^k = 1^k = 1$

Hence proved.

34. Prove that, if n is any odd integer, then $(-1)^n = -1$.

Proof: Let $n \in \mathbb{Z}$ such that n is odd.

By definition of odd integers,

$n = 2k+1$ for some $k \in \mathbb{Z}$.

$$\begin{aligned} \therefore (-1)^n &= (-1)^{2k+1} \\ &= (-1)^{2k} \cdot (-1)^1 \\ &= ((-1)^2)^k \cdot (-1) \\ &= 1^k \cdot (-1) \\ &= -1 \end{aligned}$$

Hence proved.

43 Prove that, the product of any two odd integers is odd.

(ie) to prove, $\forall m, n \in \mathbb{Z}$, if m and n are odd then mn is odd.

Proof: Let $m, n \in \mathbb{Z}$ such that m and n are odd.

By definition of odd integers,

$m = 2k_1 + 1$ for some $k_1 \in \mathbb{Z}$

$n = 2k_2 + 1$ for some $k_2 \in \mathbb{Z}$

$$\begin{aligned} \therefore mn &= (2k_1+1)(2k_2+1) \\ &= 4k_1k_2 + 2k_1 + 2k_2 + 1 \\ &= 2(2k_1k_2 + k_1 + k_2) + 1 \\ &= 2k + 1 \quad \text{where } k = 2k_1k_2 + k_1 + k_2 \end{aligned}$$

$\therefore mn$ is odd.

Hence proved.

44. Prove that the negative of any odd integer is odd.

~~Solution~~ (ie) to prove, $\forall n \in \mathbb{Z}$, if n is odd then $-n$ is odd.

Proof: Let $n \in \mathbb{Z}$ such that n is odd.

By definition of odd,

$n = 2k+1$ for some $k \in \mathbb{Z}$.

$\therefore -n = -2k-1 = -2k-1+1-1$

$= -2k-2+1$

$= 2(-k-1)+1$

$= 2k'+1$ where $k' = -k-1 \in \mathbb{Z}$

$\therefore -n$ is odd.

Hence proved.

45. Prove or Disprove, the difference of any two odd integers is odd.

(ie) to prove or disprove, $\forall m, n \in \mathbb{Z}$, if m and n are odd then $m-n$ is odd.

Solution: The given statement is false.

Counterexample: Let $m=5$ and $n=3$.

Then $m-n = 5-3 = 2$ which is even.

46. Prove or disprove, the product of any even integer and any integer is even.

~~Soln~~ (ie) to prove or disprove, $\forall m, n \in \mathbb{Z}$, if m is ~~od~~ even then mn is even.

Solution: The given statement is true.

Proof: Let $m, n \in \mathbb{Z}$ such that m is even.

By definition of even integers,

$m = 2k$ for some $k \in \mathbb{Z}$.

$\therefore mn = 2kn = 2k'$ where $k' = kn \in \mathbb{Z}$.

$\therefore mn$ is even.

Hence proved.

47. Prove or Disprove, If sum of two integers is even, then one of the summands is even.

Solution: The given statement is false.

Let $a=3$ and $b=5 \Rightarrow a+b = 3+5 = 8$

Then $a+b$ is even but a and b both are odd.

48. Prove that the difference ~~between~~ of any two even integers is even.

(ie) to prove, $\forall m,n \in \mathbb{Z}$, if m and n are even then $m-n$ is even

Proof: Let $m,n \in \mathbb{Z}$ such that m and n are even.

By definition of even integers,

$m = 2k_1$ for some $k_1 \in \mathbb{Z}$

$n = 2k_2$ for some $k_2 \in \mathbb{Z}$

$\therefore m-n = 2k_1 - 2k_2 = 2(k_1-k_2) = 2k$ where $k = k_1 - k_2 \in \mathbb{Z}$

$\therefore m-n$ is even

Hence proved.

49 Prove or disprove: The difference of any two odd integers is even.

Soln (ie) to prove or disprove: $\forall m,n \in \mathbb{Z}$, if m and n are odd then $m-n$ is even.

The given statement is true.

Proof: Let $m,n \in \mathbb{Z}$ such that m and n are odd.

By definition of odd integers,

$m = 2k_1+1$ for some $k_1 \in \mathbb{Z}$

$n = 2k_2+1$ for some $k_2 \in \mathbb{Z}$

$\therefore m-n = (2k_1+1) - (2k_2+1) = 2k_1 - 2k_2 = 2(k_1-k_2)$

$= 2k$ where $k = k_1 - k_2 \in \mathbb{Z}$

$\therefore m-n$ is even.

Hence proved.

50. Prove or disprove: $\forall\, n, m \in \mathbb{Z}$, if $n-m$ is even then $n^3 - m^3$ is even.

Solution: The given statement is true.

Proof: Let $n, m \in \mathbb{Z}$ such that $n-m$ is even.

By definition of even integers:

$n - m = 2k$ for some $k \in \mathbb{Z}$.

$\therefore n^3 - m^3 = (n-m)(n^2+nm+m^2)$

$= 2k(n^2+nm+m^2)$

$= 2k'$ where $k' = k(n^2+nm+m^2) \in \mathbb{Z}$

$\therefore n^3 - m^3$ is even.

Hence proved.

51. Prove or disprove: $\forall\, n \in \mathbb{Z}$, if n is prime then $(-1)^n = -1$.

Solution: The given statement is false.

Counterexample: Let $n = 2 \Rightarrow (-1)^n = (-1)^2 = 1$

Then n is even but $(-1)^n \neq -1$.

52. Prove or disprove: $\forall\, m \in \mathbb{Z}$ if $m > 2$ then $m^2 - 4$ is composite.

Solution: The given statement is false.

Counterexample: Let $m = 3 \Rightarrow m^2 - 4 = 3^2 - 4 = 9 - 4 = 5$

Then $m > 2$ but $m^2 - 4$ is prime.

53. ~~For~~ Prove or disprove: $\forall\, n \in \mathbb{Z}$, ~~[illegible]~~ $n^2 - n + 11$ is prime.

Solution: The given statement is false.

Counterexample: Let $n = 11 \Rightarrow n^2 - n + 11 = 11^2 - 11 + 11 = 11^2$

(ie) $n^2 - n + 11 = 11 \times 11$

~~Then~~ $\therefore n^2 - n + 11$ is composite.

54: Prove or disprove; $4(n^2+n+1)-3n^2$ is a perfect square.

Solution: The given statement is true.

Let $n \in \mathbb{Z}$.

Consider, $4(n^2+n+1)-3n^2$

$= 4n^2+4n+4-3n^2$

$= n^2+4n+4$

$= (n+2)^2$

$= k^2$ where $k = n+2 \in \mathbb{Z}$.

Hence proved.

55. Prove or disprove, every positive integer can be expressed as a sum of three or fewer perfect squares.

Solution: The given statement is false.

Counterexample:

$n=7$ cant be written as a sum of three or fewer perfect squares.

57 Prove or disprove: If m and n are positive integers and mn is a perfect square, then m and n are perfect squares.

Solution: The given statement is false.

Counterexample: Let $m=5$ and $n=5 \Rightarrow mn=25$

Then mn is a perfect square but m and n are not perfect squares.